Session 2: Introduction to continuous time economics

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Macroeconomics II

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Contents

Plan

- Why continuous time?
- Review: stochastic calculus
- From discrete time to continuous time
  1. How to conceptually bridge the two?
  2. How to mathematically define continuous time problems?
  3. How to solve continuous time problems?
- Hamilton-Jacobi-Bellman (HJB) equation
- Maximum principle
- Deterministic steady state analysis
- (Optional) Introduction to numerical methods

Why continuous time?

- The real world is continuous time
- Many advantages
  - Analytical tractability
  - Richer modeling of intertemporal decisions and uncertainty
  - Consistent with intractable complex discrete time models
- Standard in finance and many fields e.g. Merton's portfolio problem
- "Renaissance"
  - More departments are teaching continuous time in graduate even undergraduate courses
  - Distributional economics by Moll, Kaplan, and others
  - Heterogenous Agent New Keynesian (HANK) models

What mathematical tools are involved?

Looks good, but the math is a bit scary!

- Stochastic calculus
  - Brownian motion
  - Ito's lemma
  - Stochastic differential equations (SDEs)
  - ...
- Stochastic control
  - Maximum principle
  - Hamilton-Jacobi-Bellman (HJB) equation
  - Dynamic programming
  - ...
- Partial differential equations (PDE) and dynamic systems
  - 2nd order non-linear and pseudo-linear PDEs
  - Kolmogorov equations
  - ...
- Numerical PDE
  - Finite difference methods
  - Barles & Souganidis (1991) monotonic scheme analysis
  - ...

But don't worry, only a few of them are needed in this introductory session.

Reading list & expectations

Textbook & notes:

1. Benjamin Moll's notes: https://benjaminmoll.com/lectures/
2. Stokey, N. L. (2008). *The Economics of Inaction: Stochastic Control models with fixed costs*. Princeton University Press.

Suggested readings:

1. Yong, J., & Zhou, X. Y. (2012). *Stochastic controls: Hamiltonian systems and HJB equations* (Vol. 43). Springer Science & Business Media.
2. Kaplan, G., Moll, B., & Violante, G. L. (2018). Monetary policy according to HANK. *American Economic Review*, *108*(3), 697–743.
3. Brunnermeier, M. K., & Sannikov, Y. (2014). A macroeconomic model with a financial sector. *American Economic Review*, *104*(2), 379–421.
4. Glawion, R. (2023). Sequence-Space Jacobians in Continuous Time. *Available at SSRN 4504829*.

Reading list & expectations

After this session, you should be able to:

1. Understand core concepts and math formulations of continuous time economics
2. Understand standard economic models in continuous time and connect them to discrete time counterparts
3. Define economic problems in continuous time as stochastic control problems and HJB equations
4. Analytically solve the policy functions given an HJB equation
5. Analytically solve steady states (long-run equilibrium) of simple deterministic models
6. Learn where to find more information about continuous time economics

Review of stochastic calculus basics

Definition: A continuous time process $\{W(t)\}_{t\geq 0}$ taking values in $\mathbb{R}$ is called a *standard Brownian motion* (or *standard Wiener process*) if

1. $W(0) = 0$
2. $W(t) - W(s)$ is independent of the past history $\{W(r)\}_{r\leq s}$ for $0 \leq s < t$
3. $W(t) - W(s) \sim \mathcal{N}(0, t-s)$ for $0 \leq s < t$
4. $W(t)$ is continuous in t w.p.1

Multivariate Brownian motion: $W(t) = (W_1(t), W_2(t), ..., W_n(t))$ is a vector of n Brownian motions (possibly correlated)

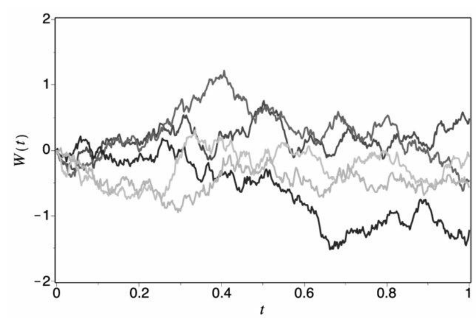


Figure 1: Sample paths of a standard Brownian motion[1]

[1]Source: Financial Mathematics: A Comprehensive Treatment in Continuous Time Volume II

Review of stochastic calculus basics

A n-dimensional stochastic process $X(t)$ is called a *diffusion process* if there exist functions $\mu(t,x)$ and $\sigma(t,x)$ such that

$$\mathrm{d}X(t) = \mu(t, X(t))\,\mathrm{d}t + \sigma(t, X(t))\,\mathrm{d}W(t)$$

where

- $W(t)$ is an m-dimensional Brownian motion
- $\mu(t, X(t)) \mapsto \mathbb{R}^n$ is a n-dimensional drift coefficient
- $\sigma(t, X(t)) \mapsto \mathbb{R}^{n\times m}$ is the diffusion/volatility coefficient

The equation is called a *stochastic differential equation* (SDE)

Properties:

- (Markovian) A diffusion process is a Markov process
- (Continuity) A diffusion process is continuous in t w.p.1

Review of stochastic calculus basics

Some commonly used processes:

- Ornstein-Uhlenbeck (OU) process: $dX(t) = \theta(\mu - X(t))\mathrm{d}t + \sigma \mathrm{d}W(t)$
  - Useful for modeling mean-reverting processes
  - Counterpart of AR(1) process: $x_{t+1} = \rho x_t + (1-\rho)\overline{x} + \varepsilon_{t+1}$
  - Solution: $X(t) = \mu + (X(0) - \mu)\exp(-\theta t) + \sigma \int_0^t \exp(-\theta(t-s))\mathrm{d}W(s)$
- Geometric Brownian motion (GBM): $\frac{dX(t)}{X(t)} = \mu \mathrm{d}t + \sigma \mathrm{d}W(t)$
  - Useful for modeling asset prices
  - Random walk with drift, non-negative
  - Solution: $X(t) = X(0)\exp\Big(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W(t)\Big)$
- Cox-Ingersoll-Ross (CIR) process: $dX(t) = \theta(\mu - X(t))\mathrm{d}t + \sigma\sqrt{X(t)}\mathrm{d}W(t)$
  - Useful for modeling interest rates
  - Non-negative, mean-reverting
- (Extra) Poisson process aka continuous-time Markov chain
  - Useful for modeling discrete state processes (e.g. unemployment)

Review of stochastic calculus basics

- **Ito's lemma** is the foundation of stochastic calculus
- Stochastic version of the chain rule in calculus
- Allows us to compute the infinitesimal change in a function of a stochastic process
- Used to derive the dynamics of a function of a diffusion process

Kiyoshi Ito (伊藤 清)
1915-2008
1970 at Cornell University

Review of stochastic calculus basics

Let $X(t)$ be a scalar diffusion process

$$dX(t) = \mu(t, X(t))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}W(t)$$

and $f(t, x)$ be a twice continuously differentiable function. Then $f(t, X(t))$ is also a diffusion process and satisfies

$$\mathrm{d}f = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\right)\mathrm{d}t + \sigma\frac{\partial f}{\partial x}\mathrm{d}W(t)$$

Useful rules:

- $\mathrm{d}t \cdot \mathrm{d}t = 0$, $\mathrm{d}t \cdot \mathrm{d}W(t) = 0$, and $\mathrm{d}W(t) \cdot \mathrm{d}W(t) = \mathrm{d}t$
- Addition rule: $\mathrm{d}(X \pm Y) = \mathrm{d}X \pm \mathrm{d}Y$
- Product rule: $\mathrm{d}(XY) = X\mathrm{d}Y + Y\mathrm{d}X + \mathrm{d}X\mathrm{d}Y$
- Quotient rule: $\mathrm{d}\left(\frac{X}{Y}\right) = \frac{\mathrm{d}X}{Y} - \frac{X\mathrm{d}Y}{Y^2} + \frac{X\mathrm{d}Y^2}{Y^3} - \frac{\mathrm{d}X\mathrm{d}Y}{Y^2}$

Review of stochastic calculus basics

Example

Let $p(t)$ be the process of house price, $h(t)$ be the process of household housing stock:

$$\frac{\mathrm{d}p(t)}{p(t)} = \mu_p \mathrm{d}t + \sigma_p \mathrm{d}W(t)$$

$$\mathrm{d}h(t) = I\mathrm{d}t$$

What is the SDE of the housing value $V(t) := p(t)h(t)$?

$$\begin{aligned} &\mathrm{d}V(t) = p(t)\mathrm{d}h(t) + h(t)\mathrm{d}p(t) + \mathrm{d}p(t)\mathrm{d}h(t) \\ &= p(t)\cdot[I\mathrm{d}t] + h(t)\cdot\left[\mu_p p(t)\mathrm{d}t + \sigma_p p(t)\mathrm{d}W(t)\right] + [I\mathrm{d}t]\cdot\left[\mu_p p(t)\mathrm{d}t + \sigma_p p(t)\mathrm{d}W(t)\right] \\ &= \left\{p(t)\cdot I\mathrm{d}t + \mu_p h(t)p(t)\mathrm{d}t + h(t)\sigma_p p(t)\mathrm{d}W(t)\right\} + \left\{I\mu_p p(t)\mathrm{d}t\cdot\mathrm{d}t + I\sigma_p p(t)\mathrm{d}t\cdot\mathrm{d}W(t)\right\} \\ &= \left[p(t)\cdot I + \mu_p V(t)\right]\mathrm{d}t + \sigma_p V(t)\mathrm{d}W(t) + \{0 + 0\} \end{aligned}$$

Review of stochastic calculus basics

Multivariate Ito's lemma:

Let

$$dX(t) = \mu(t, X(t))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}W(t)$$

$$X(t) \in \mathbb{R}^n, W(t) \in \mathbb{R}^m, \mu \mapsto \mathbb{R}^n, \sigma \mapsto \mathbb{R}^{n\times m}$$

and $f(t, x) \mapsto \mathbb{R}$ is a twice continuously differentiable function. Then

$$df(t, X(t)) = \left\{\frac{\partial f}{\partial t} + (\nabla_X f)^T \mathrm{d}X(t) + \frac{1}{2}(\mathrm{d}X(t))^T(\nabla_X^2 f)\mathrm{d}X(t)\right\}\mathrm{d}t + (\nabla_X f)^T \sigma \mathrm{d}B(t)$$

where

- $\nabla_X f \in \mathbb{R}^n$ is the gradient of f with respect to X
- $\nabla_X^2 f \in \mathbb{R}^{n\times n}$ is the Hessian of f with respect to X

- Useful in models with multiple uncertainties. FYI, no need to memorize

From discrete time to continuous time

In macroeconomic models, we usually care lifetime problems like:

$$v_0(x_0) = \max_{\{c_t\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{T} \beta^t u(t, c_t, x_t) + f(x_T) \right\}$$

$$\text{s.t.} \quad x_{t+1} = g(t, x_t, c_t)$$

$$c_t \in \mathbb{C}_t, x_t \in \mathbb{X}_t$$

- x_0 is the initial state
- x is the state vector (e.g. capital, asset)
- c is the control vector (e.g. consumption, investment)
- $u(t, c, x)$ is the instantaneous utility function
- $f(x)$ is the terminal utility function
- $g(t, x, c)$ is the state equation or law of motion (e.g. budget constraint)
- $\mathbb{C}_t$ is the set of admissible controls, and $\mathbb{X}_t$ is the set of admissible states (state constraints)

From discrete time to continuous time

Example: Neo-classical growth model

$$v_0(k_0) = \max_{\{c_t\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \log(c_t) \right\}$$

$$\text{s.t.} \quad \forall t, k_{t+1} = (1-\delta)k_t + k_t^{\alpha} - c_t$$

$$c_t \geq 0, k_t \geq 0$$

where

- planning horizon: $T = \infty$
- state variables: k_t (capital)
- control variables: c_t (consumption)
- instantaneous utility: $\log(c)$
- terminal utility: $f(x) = \lim_{t\to\infty} \beta^t \log(c_t) = 0$ (transversality condition)
- Admissible sets
  - $\mathbb{C}_t = \{c_t \geq 0\}$
  - $\mathbb{X}_t = \{k_t \geq 0\}$

From discrete time to continuous time

By Bellman's principle, we can rewrite the problem as the following *dynamic programming* (DP) problem (aka *Bellman equation*):

$$v(t,x) = \max_{c_t} \mathbb{E}_t\{u(t,c_t,x_t) + \beta v(t+1,x_{t+1})\}$$
$$\text{s.t.} \quad x_{t+1} = g(t,x_t,c_t); c_t \in \mathbb{C}_t, x_t \in \mathbb{X}_t$$

If the problem is *time homogeneous*[1], we can drop the time index:

$$v(x) = \max_{c} \mathbb{E}\{u(c,x) + \beta v(x')\}$$
$$\text{s.t.} \quad x' = g(x,c); c \in \mathbb{C}, x \in \mathbb{X}$$

Example: Neo-classical growth model

$$v(k) = \max_{c} \mathbb{E}\{\log(c) + \beta v(k')\}$$
$$\text{s.t.} \quad k' = (1-\delta)k + k^{\alpha} - c$$
$$c \geq 0, k \geq 0$$

[1] i.e. conditions do not depend on t or the history of the states.

From discrete time to continuous time

Question: discrete time $\longrightarrow$ continuous time, how?

$$
\begin{aligned}
& v_0(x_0) = \max_{\{c_t\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{T} \beta^t u(t, c_t, x_t) + f(x_T) \right\} && \to ? \\
& \text{s.t.} \quad x_{t+1} = g(t, x_t, c_t) && \to ? \\
& c_t \in \mathbb{C}_t, x_t \in \mathbb{X}_t && \to ?
\end{aligned}
$$

Intuitively, everything should be continuous in time:

- Sequence to process: $\{x_t\}_{t=0}^{T} \longrightarrow \{x(t)\}_{t\geq 0}^{T}$, or more often $x(\cdot)$
- Summation of the infinitesimal: $\sum_{t=0}^{T} u_t \longrightarrow \int_0^T u(t, c(t), x(t))$
- State equation as stochastic process (typically an SDE):

$$
x_{t+1} = g(t, x_t, c_t) \;\longrightarrow\; \mathrm{d}x(t) = g(t, x(t), c(t))
$$

- In this session, we only discuss the case where $x(t)$ is a diffusion process, i.e. $g(t, x, c) = \mu(t, x, c)\mathrm{d}t + \sigma(t, x, c)\mathrm{d}W(t)$

From discrete time to continuous time

Then generally, we can write the continuous time problem as the following *stochastic control problem*:

$$v_0(x_0) = \max_{c(\cdot)} \mathbb{E}_0 \left\{ \int_0^T e^{-\rho t} u(t, c(t), x(t)) \mathrm{d}t + f(x(T)) \right\}$$

$$\text{s.t.} \quad \mathrm{d}x(t) = \mu(t, x(t), c(t))\mathrm{d}t + \sigma(t, x(t), c(t))\mathrm{d}W(t)$$

$$c(t) \in \mathbb{C}_t, x(t) \in \mathbb{X}_t$$

where

- ρ is the discount rate, counterpart of β in discrete time
- $x(\cdot)$ is the *(controlled) state process*, $\forall t, x(t) \in \mathbb{R}^n$
- $c(\cdot)$ is the *control process*, $\forall t, c(t) \in \mathbb{R}^k$
- $W(\cdot)$ is the *Brownian motion*, $\forall t, W(t) \in \mathbb{R}^m$
- $\mu(t, x, c) \mapsto \mathbb{R}^n$ is the drift coefficient
- $\sigma(t, x, c) \mapsto \mathbb{R}^{n \times m}$ is the diffusion/volatility coefficient

From discrete time to continuous time

Example: Neo-classical growth model

Discrete time:

$$v_0(k_0) = \max_{\{c_t, k_{t+1}\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \log(c_t) \right\}$$

$$\text{s.t.} \quad \forall t, k_{t+1} = (1-\delta)k_t + k_t^\alpha - c_t$$

$$c_t \geq 0, k_t \geq 0$$

Continuous time:

$$v_0(k_0) = \max_{c(\cdot)} \mathbb{E}_0 \left\{ \int_0^{\infty} e^{-\rho t} \log(c(t)) \mathrm{d}t \right\}$$

$$\text{s.t.} \quad \mathrm{d}k(t) = -\delta k(t) + k(t)^\alpha - c(t)$$

$$c(t) \geq 0, k(t) \geq 0$$

Notice:

- continuous discounting
- change of the control variables
- the state equation tells the *change* of $k(t)$ now
- the transversality condition: $\lim_{t\to\infty} e^{-\rho t} \log(c(t)) = 0$
- the u, c are interpreted as flow/rate rather than amount, cp. c_t in discrete time and $\int_t^{t+\Delta t} c(t)\mathrm{d}t$ in continuous time

From discrete time to continuous time

Practice 1: write the following household consumption-leisure problem in continuous time:

$$v_0(a_0) = \max_{\{c_t, a_{t+1}, n_t\}_t} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma}}{1-\gamma} - \frac{n_t^{1+\nu}}{1+\nu} \right]$$

$$\text{s.t. } \; c_t + a_{t+1} = (1+r)a_t + wn_t$$

From discrete time to continuous time

Practice 1: write the following household consumption-leisure problem in continuous time:

$$v_0(a_0) = \max_{\{c_t, a_{t+1}, n_t\}_t} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma}}{1-\gamma} - \frac{n_t^{1+\nu}}{1+\nu} \right]$$

$$\text{s.t.} \quad c_t + a_{t+1} = (1+r)a_t + wn_t$$

Continuous time version:

$$v_0(a_0) = \max_{c(\cdot), n(\cdot)} \int_0^{\infty} e^{-\rho t} \left[\frac{c(t)^{1-\gamma}}{1-\gamma} - \frac{n(t)^{1+\nu}}{1+\nu} \right] \mathrm{d}t$$

$$\text{s.t.} \quad \mathrm{d}a(t) = \{ra(t) + wn(t) - c(t)\}\mathrm{d}t$$

From discrete time to continuous time

Practice 2: write the following entrepreneur's problem in continuous time:

$$v_0(k_0, z_0) = \max_{\{c_t, k_{t+1}, n_t\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \log(c_t) \right\}$$

$$\text{s.t.} \quad c_t + k_{t+1} = z_t k_t^{\alpha} n_t^{1-\alpha} - w n_t$$

$$z_t \sim \text{AR}(1) \text{ process}$$

Hint: use OU process $\mathrm{d}z(t) = \theta(\mu - z(t))\mathrm{d}t + \sigma \mathrm{d}W(t)$

From discrete time to continuous time

Practice 2: write the following entrepreneur's problem in continuous time:

$$v_0(k_0, z_0) = \max_{\{c_t, k_{t+1}, n_t\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \log(c_t) \right\}$$

$$\text{s.t.} \quad c_t + k_{t+1} = z_t k_t^{\alpha} n_t^{1-\alpha} - w n_t$$

$$z_t \sim \text{AR}(1) \text{ process}$$

Hint: use OU process $\mathrm{d}z(t) = \theta(\mu - z(t))\mathrm{d}t + \sigma \mathrm{d}W(t)$

Continuous time version:

$$v_0(k_0, z_0) = \max_{c(\cdot), n(\cdot)} \mathbb{E}_0 \left\{ \int_0^{\infty} e^{-\rho t} \log(c(t)) \mathrm{d}t \right\}$$

$$\text{s.t.} \quad \mathrm{d}k(t) = \{z(t)k(t)^{\alpha} n(t)^{1-\alpha} - w n(t) - c(t)\} \mathrm{d}t$$

$$\mathrm{d}z(t) = \theta(\mu - z(t))\mathrm{d}t + \sigma \mathrm{d}W(t)$$

From discrete time to continuous time

Practice 3: write the following price setter's problem with Rotemberg price rigidity in continuous time:

$$v_0(p_0, y_0) = \max_{\{p_{t+1}\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t y_t \cdot \left[p_t - \frac{\varphi}{2} (p_{t+1} - p_t)^2 \right] \right\}$$

$$\text{s.t.} \quad y_t \sim \text{AR}(1) \text{ process}$$

Hint:

- denote the drift term of $p(t)$ as $\dot{p}(t)$
- the price adjustment cost $\frac{\varphi}{2}(p_{t+1} - p_t)^2 \longrightarrow \frac{\varphi}{2}(\dot{p}(t))^2$
- use OU process $\frac{\mathrm{d}y(t)}{y(t)} = \theta(\mu - 1)\mathrm{d}t + \sigma \mathrm{d}W(t)$

From discrete time to continuous time

Practice 3: write the following price setter's problem with Rotemberg price rigidity in continuous time:

$$v_0(p_0, y_0) = \max_{\{p_{t+1}\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t y_t \cdot \left[p_t - \frac{\varphi}{2} \left(p_{t+1} - p_t\right)^2 \right] \right\}$$

$$\text{s.t.} \quad y_t \sim \text{AR(1) process}$$

Hint:

- denote the drift term of $p(t)$ as $\dot{p}(t)$
- the price adjustment cost $\frac{\varphi}{2}\left(p_{t+1} - p_t\right)^2 \longrightarrow \frac{\varphi}{2}(\dot{p}(t))^2$
- use OU process $\frac{\mathrm{d}y(t)}{y(t)} = \theta(\mu - 1)\mathrm{d}t + \sigma \mathrm{d}W(t)$

$$v_0(p_0, y_0) = \max_{\dot{p}(\cdot)} \mathbb{E}_0 \left\{ \int_0^{\infty} e^{-\rho t} y(t) \cdot \left[p(t) - \frac{\varphi}{2} (\dot{p}(t))^2 \right] \mathrm{d}t \right\}$$

$$\text{s.t.} \quad \mathrm{d}p(t) = \dot{p}(t)\mathrm{d}t$$

$$\mathrm{d}y(t) = \theta(\mu - 1)\mathrm{d}t + \sigma \mathrm{d}W(t)$$

Dynamic programming in continuous time

- Bellman's principle applies to the continuous time case as well

$$v(t,x) = \max_{c_t} \mathbb{E}_t \{u(t,c_t,x_t) + \beta v(t+1,x_{t+1})\}$$
$$\text{s.t.} \quad x_{t+1} = g(t,x_t,c_t); c_t \in \mathbb{C}_t, x_t \in \mathbb{X}_t$$

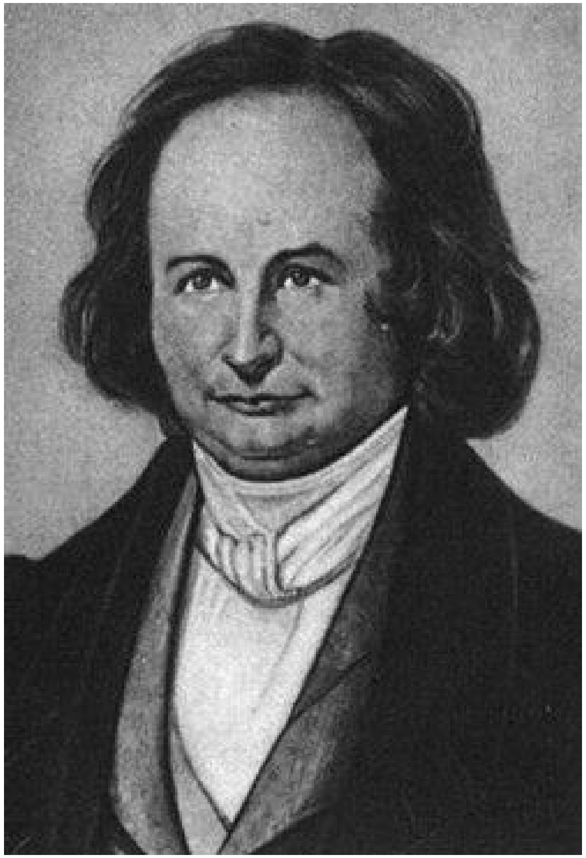
$$\downarrow\downarrow\downarrow$$

$$v(t,x) = \max_{c_{[t,t+\Delta t)}} \mathbb{E}_t \left\{ \int_t^{t+\Delta t} e^{-\rho s} u(s,c(s),x(s)) \mathrm{d}s + e^{-\rho \Delta t} v(t+\Delta t, x(t+\Delta t)) \right\}$$
$$\text{s.t.} \quad \mathrm{d}x(t) = \mu(t,x(t),c(t))\mathrm{d}t + \sigma(t,x(t),c(t))\mathrm{d}W(t)$$
$$c(t) \in \mathbb{C}_t, x(t) \in \mathbb{X}_t$$

- $\Delta t > 0$ is the time increment
- Can be solved exactly by the same method as in discrete time: value function iteration, policy function iteration, ...
- Smaller Δt, closer to the true solution
- But what if $\Delta t \to 0$? what math tools are needed?

HJB equation

William Hamilton



Carl Jacobi

Richard Bellman

Hamilton-Jacobi-Bellman (HJB) equation

HJB equation

- By expanding the continuous time Bellman equation[1], we can derive the following *Hamilton-Jacobi-Bellman (HJB) equation*:

$$\rho v(t,x) = \max_c u(t,c,x) + \frac{\partial v}{\partial t} + \mu(t,x,c)^T \cdot \nabla_x v + \frac{1}{2}\operatorname{tr}(\sigma(t,x,c)\sigma(t,x,c)^T \nabla_x^2 v)$$

$$\text{s.t.} \quad c \in \mathbb{C}_t, x \in \mathbb{X}_t$$

By defining the *2nd-order infinitesimal generator*

$$\mathcal{L}[v] := \frac{\partial v}{\partial t} + \mu(t,x,c)^T \cdot \nabla_x v + \frac{1}{2}\operatorname{tr}(\sigma(t,x,c)\sigma(t,x,c)^T \nabla_x^2 v)$$

The HJB equation is sometimes written as the following *variational inequality*:

$$\max\{u(t,c,x) + \mathcal{L}[v] - \rho v(t,x)\} = 0$$

[1]Chapter 4.3 of Yong & Zhou (2012), FYI

HJB equation

$$\rho v(t,x) = \max_c u(t,c,x) + \frac{\partial v}{\partial t} + \mu(t,x,c)^T \cdot \nabla_x v + \frac{1}{2}\,\mathrm{tr}(\sigma(t,x,c)\sigma(t,x,c)^T \nabla_x^2 v)$$

Tips

- $\mu(t,x,c)^T \cdot \nabla_x v \in \mathbb{R}$, inner product of the state drifts and the gradients of the value function
- $\sigma(t,x,c)\sigma(t,x,c)^T \nabla_x^2 v \in \mathbb{R}^{n\times n}$, the covariance matrix times the Hessian of the value function
- $\frac{\partial v}{\partial t}$, the time derivative of the value function, ignored in time homogeneous problems where t can be ignored[1]
- The red term, in some contexts, is called the *flux term* which tells the mean or deterministic part of how the value function changes over time
- the blue term is called the *diffusion term* which tells the *risk adjustment* of the value function due to the uncertainty

[1] $\mathrm{d}t = 1\mathrm{d}t$, thus $1 \cdot \frac{\partial v}{\partial t}$

HJB equation

$$\rho v(t,x) = \max_c u(t,c,x) + \frac{\partial v}{\partial t} + \mu(t,x,c)^T \cdot \nabla_x v + \frac{1}{2}\operatorname{tr}(\sigma(t,x,c)\sigma(t,x,c)^T \nabla_x^2 v)$$

Compared with the discrete time Bellman equation:

- No expectation operator → all expectations about the future's uncertainty happens in the infinitesimal small time increment dt and summarized by the infinitesimal generator $\mathcal{L}[v]$ *explicitly*
  - which is impossible to do in discrete time
- split of the risk adjustment
- HJB equation is a special kind of 2nd-order non-linear partial differential equation (PDE) in t and x (state variables)
  - elliptic (**椭圆**) if time homogeneous
  - parabolic (**抛物线**) if time non-homogeneous

HJB equation

Example: Stochastic neo-classical growth model

$$v(0,k(0),z(0)) = \max_{c(\cdot)} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} \log(c(t)) \mathrm{d}t \right\}$$

$$\text{s.t.} \quad \mathrm{d}k(t) = -\delta k(t) + z(t)k(t)^\alpha - c(t)$$

$$\mathrm{d}z(t) = \theta(\mu - z(t))\mathrm{d}t + \sigma \mathrm{d}W(t)$$

$$\downarrow\downarrow\downarrow$$

$$v(t,k(t),z(t)) = \max_{c_{[t,t+\Delta t)}} \mathbb{E}_t \left\{ \int_t^{t+\Delta t} e^{-\rho s} \log(c(s))\mathrm{d}s + e^{-\rho \Delta t} v(t+\Delta t, k(t+\Delta t)) \right\}$$

$$\text{s.t.} \quad \mathrm{d}k(t) = (-\delta k(t) + k(t)^\alpha - c(t))\mathrm{d}t$$

$$\mathrm{d}z(t) = \theta(\mu - z(t))\mathrm{d}t + \sigma \mathrm{d}W(t)$$

$$\downarrow\downarrow\downarrow$$

$$\rho v(t,k(t),z(t)) = \max_c \log(c) + \frac{\partial v}{\partial t} + (-\delta k(t) + k(t)^\alpha - c(t)) \cdot \frac{\partial v}{\partial k} + \theta(\mu - z(t)) \cdot \frac{\partial v}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial k^2}$$

$$\downarrow\downarrow\downarrow \text{ (time homogeneity)}$$

$$\rho v(k,z) = \max_c \log(c) + (-\delta k + k^\alpha - c) \cdot \frac{\partial v}{\partial k} + \theta(\mu - z) \cdot \frac{\partial v}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial z^2}$$

HJB equation

Practice 1: write the HJB equation for the following household consumption-leisure problem in continuous time:

$$\max_{c(\cdot),n(\cdot)} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} \left[\frac{c(t)^{1-\gamma}}{1-\gamma} - \frac{n(t)^{1+\nu}}{1+\nu} \right] \mathrm{d}t \right\}$$

$$\text{s.t.} \quad \mathrm{d}a(t) = \{ra(t) + w(t)n(t) - c(t)\}\mathrm{d}t$$

$$\mathrm{d}w(t) = \theta(\mu - w(t))\mathrm{d}t + \sigma \mathrm{d}W(t)$$

Hint: time homogeneous (does not depend on t or the history of the states)

HJB equation

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$$\text{s.t.} \quad \mathrm{d}a(t) = \{ra(t) + w(t)n(t) - c(t)\}\mathrm{d}t$$

$$\mathrm{d}w(t) = \theta(\mu - w(t))\mathrm{d}t + \sigma \mathrm{d}W(t)$$

Hint: time homogeneous (does not depend on t or the history of the states)

HJB equation:

$$\rho v(a,w) = \max_{c,n} \frac{c^{1-\gamma}}{1-\gamma} - \frac{n^{1+\nu}}{1+\nu} + \frac{\partial v}{\partial a} \cdot (ra + wn - c) + \frac{\partial v}{\partial w} \cdot (\theta(\mu - w)) + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial w^2}$$

HJB equation

Practice 2: write the HJB equation for the following price setter's problem with Rotemberg price rigidity in continuous time:

$$v_0(p_0, y_0) = \max_{\dot{p}(\cdot)} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} y(t) \cdot \left[p(t) - \frac{\varphi}{2} (\dot{p}(t))^2 \right] \mathrm{d}t \right\}$$

$$\text{s.t.} \quad \mathrm{d}p(t) = \dot{p}(t)\mathrm{d}t$$

$$\mathrm{d}y(t) = \theta(\mu - 1)\mathrm{d}t + \sigma \mathrm{d}W(t)$$

HJB equation

Practice 2: write the HJB equation for the following price setter's problem with Rotemberg price rigidity in continuous time:

$$v_0(p_0, y_0) = \max_{\dot{p}(\cdot)} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} y(t) \cdot \left[p(t) - \frac{\varphi}{2} (\dot{p}(t))^2 \right] \mathrm{d}t \right\}$$

$$\text{s.t.} \quad \mathrm{d}p(t) = \dot{p}(t)\mathrm{d}t$$

$$\mathrm{d}y(t) = \theta(\mu - 1)\mathrm{d}t + \sigma \mathrm{d}W(t)$$

HJB equation:

$$\rho v(p, y) = \max_{\dot{p}} y \cdot \left[p - \frac{\varphi}{2} (\dot{p})^2 \right] + \frac{\partial v}{\partial p} \cdot \dot{p} + \frac{\partial v}{\partial y} \cdot (\theta(\mu - 1)) + \frac{1}{2} \sigma^2 \frac{\partial^2 v}{\partial y^2}$$

HJB equation

Practice 3: write the HJB equation for the following homeowner's problem:

$$v_0(a_0, h_0, z_0) = \max_{c(\cdot), I(\cdot), n(\cdot)} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} \left[\frac{\left[c(t)^\alpha h(t)^{1-\alpha}\right]^{1-\gamma}}{1-\gamma} - \frac{n(t)^{1+\nu}}{1+\nu} \right] \mathrm{d}t \right\}$$

$$\text{s.t.} \quad \mathrm{d}a(t) = \left\{ ra(t) + z(t)n(t) - c(t) - \left[I(t) + \frac{\psi}{2} I(t)^2 \right] p \cdot h(t) \right\} \mathrm{d}t$$

$$\mathrm{d}h(t) = I(t)\mathrm{d}t$$

$$\mathrm{d}z(t) = \theta(\mu - z(t))\mathrm{d}t + \sigma \mathrm{d}W(t)$$

HJB equation

Practice 3: write the HJB equation for the following homeowner's problem:

$$v_0(a_0,h_0,z_0) = \max_{c(\cdot),I(\cdot),n(\cdot)} \mathbb{E}_0\left\{\int_0^\infty e^{-\rho t}\left[\frac{\left[c(t)^\alpha h(t)^{1-\alpha}\right]^{1-\gamma}}{1-\gamma} - \frac{n(t)^{1+\nu}}{1+\nu}\right]\mathrm{d}t\right\}$$

$$\text{s.t.}\quad \mathrm{d}a(t) = \left\{ra(t) + z(t)n(t) - c(t) - \left[I(t) + \frac{\psi}{2}I(t)^2\right]p\cdot h(t)\right\}\mathrm{d}t$$

$$\mathrm{d}h(t) = I(t)\mathrm{d}t$$

$$\mathrm{d}z(t) = \theta(\mu - z(t))\mathrm{d}t + \sigma \mathrm{d}W(t)$$

HJB equation:

$$\rho v(a,h,z) = \max_{c,I,n} \frac{\left[c^\alpha h^{1-\alpha}\right]^{1-\gamma}}{1-\gamma} - \frac{n^{1+\nu}}{1+\nu}$$

$$+\frac{\partial v}{\partial a}\cdot\left(ra + zn - c - \left[I + \frac{\psi}{2}I^2\right]p\cdot h\right) + \frac{\partial v}{\partial h}\cdot I + \frac{\partial v}{\partial z}\cdot(\theta(\mu - z)) + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial z^2}$$

HJB equation

A more general example:

$$\rho v(x) = \max_c u(c) + \nabla_x v \cdot \mu + \frac{1}{2} \operatorname{tr}[\sigma \cdot \sigma^T \cdot \nabla_x^2 v]$$

$$\text{s.t.} \quad \mathrm{d}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \mathrm{d}t + \underbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \end{bmatrix}}_{2\times 3} \mathrm{d}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}$$

Notice: the uncertainty/risk are correlated

The flux term:

$$\nabla_x v \cdot \mu = \frac{\partial v}{\partial x_1}\mu_1 + \frac{\partial v}{\partial x_2}\mu_2$$

The diffusion term:

$$\frac{1}{2}\operatorname{tr}(\sigma\cdot\sigma^T\cdot\nabla_x^2 v)=\frac{1}{2}\operatorname{tr}\left\{\underbrace{\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\end{bmatrix}}_{2\times 3}\cdot\underbrace{\begin{bmatrix}\sigma_{11} & \sigma_{21}\\ \sigma_{12} & \sigma_{22}\\ \sigma_{13} & \sigma_{23}\end{bmatrix}}_{3\times 2}\cdot\underbrace{\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}}_{2\times 2}\right\}$$

$$=\frac{1}{2}\operatorname{tr}\left\{\underbrace{\begin{bmatrix}\sum_j \sigma_{1j}^2 & \sum_j \sigma_{1j}\sigma_{2j}\\ \sum_j \sigma_{2j}\sigma_{1j} & \sum_j \sigma_{2j}^2\end{bmatrix}}_{\text{covariance matrix}}\cdot\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}\right\}$$

$$=\frac{1}{2}\operatorname{tr}\left[\begin{array}{c|c}\color{red}{v_{11}\cdot\sum_j \sigma_{1j}^2+v_{21}\cdot\sum_j \sigma_{1j}\sigma_{2j}} & v_{12}\cdot\sum_j \sigma_{1j}^2+v_{22}\cdot\sum_j \sigma_{1j}\sigma_{2j}\\ \hline v_{11}\cdot\sum_j \sigma_{2j}\sigma_{1j}+v_{21}\cdot\sum_j \sigma_{2j}^2 & \color{red}{v_{12}\cdot\sum_j \sigma_{2j}\sigma_{1j}+v_{22}\cdot\sum_j \sigma_{2j}^2}\end{array}\right]$$

$$=\frac{1}{2}\left\{\underbrace{\left[v_{11}\cdot\sum_j \sigma_{1j}^2+v_{22}\cdot\sum_j \sigma_{2j}^2\right]}_{\text{risk adjustment due to Var}}+\underbrace{\left[v_{21}\cdot\sum_j \sigma_{1j}\sigma_{2j}+v_{12}\cdot\sum_j \sigma_{2j}\sigma_{1j}\right]}_{\text{risk adjustment due to Cov (systematic risk)}}\right\}$$

- Common in finance models and models caring about systematic risk
- Asymmetric covariance allowed

HJB equation

(*n-independent risks*) However, most models in macro fits into the following special forms:

$$\mathrm{d}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} \mathrm{d}t + \underbrace{\begin{bmatrix} \sigma_{11} & & \\ & \ddots & \\ & & \sigma_{nn} \end{bmatrix}}_{2\times 3} \mathrm{d}\begin{bmatrix} W_1 \\ \vdots \\ W_n \end{bmatrix}$$

$$\downarrow\downarrow\downarrow$$

$$\frac{1}{2}\,\mathrm{tr}\left[\sigma \cdot \sigma^T \cdot \nabla_x^2 v\right] = \frac{1}{2}\sum_{i=1}^{n} \sigma_{ii}^2 \cdot v_{ii}$$

Implications:

- Every state can be affected by at most one risk source
- In GE, all risks can be expressed as a linear combination of the “final” risk sources (e.g. TFP shock)

Solving policy functions

To solve the policy functions, one can simply deriving the first order condition (FOC) of the RHS of the HJB equation.

Example: stochastic neo-classical growth model

$$\rho v(k,z) = \max_{c} \log(c) + (-\delta k + z k^{\alpha} - c) \cdot \frac{\partial v}{\partial k} + \theta(\mu - z) \cdot \frac{\partial v}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial z^2}$$

Define objective function:

$$\mathcal{H}\left(c; k, z; \nabla_{k,z} v; \nabla^2_{k,z} v\right) := \log(c) + (-\delta k + z k^{\alpha} - c) \cdot \frac{\partial v}{\partial k} + \theta(\mu - z) \cdot \frac{\partial v}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial z^2}$$

FOC:

$$\frac{\partial \mathcal{H}}{\partial c} = \frac{1}{c} - \frac{\partial v}{\partial k} = 0$$

$$\implies c = \left(\frac{\partial v}{\partial k}\right)^{-1}$$

Solving policy functions

Compare with the discrete time case:

$$v(k,z) = \max_c \log(c) + \beta \mathbb{E} v(k', z')$$
$$\text{s.t.} \quad c + k' = (1-\delta)k + zk^\alpha$$
$$z \sim \text{AR}(1) \text{ process}$$

Lagrangian:

$$\mathcal{L}(c;k,z) := \log(c) + \beta \mathbb{E} v(k', z') + \lambda((1-\delta)k + zk^\alpha - c - k')$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial c} = \frac{1}{c} - \lambda = 0$$
$$\implies c = \lambda^{-1}$$

- The Lagrangian multiplier λ is the *shadow value*
- In continuous time, the shadow value is $\frac{\partial v}{\partial k}$

Solving policy functions

Practice 1: solve the optimal inflation rate for the following price setter's problem with Rotemberg price rigidity in continuous time:

$$\rho v(p, y) = \max_{\pi} py \cdot \left(1 - \frac{\varphi}{2}\pi^2\right) + \frac{\partial v}{\partial p} \cdot \pi p + \frac{\partial v}{\partial y} \cdot \theta(\overline{y} - y) + \frac{1}{2}\sigma^2 y \frac{\partial^2 v}{\partial y^2}$$

Solving policy functions

Practice 1: solve the optimal inflation rate for the following price setter's problem with Rotemberg price rigidity in continuous time:

$$\rho v(p,y) = \max_{\pi} py \cdot \left(1 - \frac{\varphi}{2}\pi^2\right) + \frac{\partial v}{\partial p} \cdot \pi p + \frac{\partial v}{\partial y} \cdot \theta(\overline{y} - y) + \frac{1}{2}\sigma^2 y \frac{\partial^2 v}{\partial y^2}$$

FOC:

$$\frac{\partial \mathcal{H}}{\partial \pi} = -\varphi p y \pi + p \frac{\partial v}{\partial p} = 0$$

$$\implies \pi = \varphi y \left(\frac{\partial v}{\partial p}\right)^{-1}$$

Solving policy functions

Practice 2 Solve the policy function for the following household problem with a liquid asset a and an illiquid asset h:

$$\rho v(a,h,z) = \max_{c,I,n} \frac{\left[c^{\alpha} h^{1-\alpha}\right]^{1-\gamma}}{1-\gamma} - \frac{n^{1+\nu}}{1+\nu}$$

$$+\frac{\partial v}{\partial a} \cdot \left(ra + zn - c - \left[I + \frac{\psi}{2} I^2\right] p \cdot h\right) + \frac{\partial v}{\partial h} \cdot Ih + \frac{\partial v}{\partial z} \cdot \theta(\mu - z) + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial z^2}$$

Solving policy functions

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$$+\frac{\partial v}{\partial a} \cdot \left(ra + zn - c - \left[I + \frac{\psi}{2} I^2\right] p \cdot h\right) + \frac{\partial v}{\partial h} \cdot Ih + \frac{\partial v}{\partial z} \cdot \theta(\mu - z) + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial z^2}$$

FOC:

$$\frac{\partial \mathcal{H}}{\partial c} = \alpha h^{(1-\alpha)(1-\gamma)} c^{\alpha(1-\gamma)-1} - \frac{\partial v}{\partial a} = 0$$

$$\frac{\partial \mathcal{H}}{\partial I} = -\frac{\partial v}{\partial a} \cdot [1 + \psi I] ph + \frac{\partial v}{\partial h} h = 0$$

$$\frac{\partial \mathcal{H}}{\partial n} = -n^{\nu} + \frac{\partial v}{\partial a} z = 0$$

Solving policy functions

which implies:

$$c = \left(\frac{1}{\alpha}\frac{\partial v}{\partial a}\right)^{\frac{1}{\alpha(1-\gamma)-1}} h^{\frac{(1-\alpha)(1-\gamma)}{\alpha(1-\gamma)-1}}$$

$$I = \frac{1}{\psi}\left[\frac{\partial v/\partial h}{\partial v/\partial a}\frac{1}{p} - 1\right]$$

$$n = \left(\frac{\partial v}{\partial a} z\right)^{\frac{1}{\nu}}$$

- The level of consumption c is *static*: only depends on the current value function and the current state h; as well as I and n
- However, consumption is *dynamic* in nature: charaterized by Euler equation $\Longrightarrow$ center in understanding the macroeconomic intuitions

Solving policy functions

Dynamic analysis is the center of macroeconomic models

- Long-run equilibrium (steady state)
- Inter-temporal trade-off
- Transition dynamics
- ...

This is done by caring the dynamics of:

- state variables $x(t)$
- shadow values $\lambda(t)$
- control variables $c(t)$ (Euler equation)

Solving policy functions

- Check the textbook example of the growth model in discrete time:

$$v(k, z) = \max_{c,k'} \log(c) + \beta \mathbb{E}\{v(k', z')|z\}$$
$$\text{s.t.} \quad c + k' = (1-\delta)k + zk^{\alpha}$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial c} = \frac{1}{c} - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial k'} = \beta \mathbb{E} \frac{\partial v'}{\partial k'} - \lambda = 0$$
$$\frac{\partial v'}{\partial k'} = \lambda' \left[(1-\delta) + \alpha z' (k')^{\alpha - 1}\right] \text{ (envelope theorem)}$$

Euler equation: $\frac{1}{c} = \beta \mathbb{E}\left\{\frac{1}{c'}\left[1 - \delta + \alpha z'(k')^{\alpha-1}\right]\right\}$

$\Longrightarrow$ How to analyze the *dynamics* of a continuous time model? e.g.

$$\mathrm{d}c(t) = ? \qquad \mathrm{d}(\partial v / \partial k) = ?$$

Dynamic analysis in continuous time

Recall the generic discrete time model:

$$v(t,x) = \max_{c_t} \mathbb{E}_t\{u(t,c_t,x_t) + \beta v(t+1,x_{t+1})\}$$

$$\text{s.t.}\quad x_{t+1} = g(t,x_t,c_t); c_t \in \mathbb{C}_t, x_t \in \mathbb{X}_t$$

Dynamics of the model includes the dynamics of:

- state variables $x_t \Longrightarrow$ state equation
- shadow values $\lambda_t \Longrightarrow$ envelope theorem
- control variables $c_t \Longrightarrow$ Euler equation

We typically do:

$$\begin{cases}\text{FOC}\\ \text{Envelope theorem}\end{cases} \Longrightarrow \text{Euler equation}$$

With the dynamics of the model: inter-temporal trade-off; long-run equilibrium (steady state); ...

Dynamic analysis in continuous time

In a continuous time model:

$$\rho v(t,x) = \max_c u(t,c,x) + \frac{\partial v}{\partial t} + \mu(t,x,c)^T \cdot \nabla_x v + \frac{1}{2}\operatorname{tr}(\sigma(t,x,c)\sigma(t,x,c)^T \nabla_x^2 v)$$

Dynamics of the model includes the dynamics of:

- state variables $x(t) \Longrightarrow$ state equation $\mathrm{d}x(t)$ ✅
- shadow values $\nabla_x v \Longrightarrow$ envelope theorem $\mathrm{d}\frac{\partial v}{\partial x_i}$ ❓
- control variables $c(t) \Longrightarrow$ Euler equation $\mathrm{d}c(t)$ ❓

$$\begin{cases}\text{FOC} \\ \text{Envelope theorem ?}\end{cases} \Longrightarrow \text{Euler equation}$$

$\Longrightarrow$ *Maximum principle*[1]

[1] The continuous time version of the envelope theorem is available but not easy to derive for the stochastic case, while the maximum principle is much easier to follow.

Dynamic analysis in continuous time

- *Maximum principle*, or Pontryagin's maximum principle (PMP), is a necessary condition for optimality in continuous time control problems

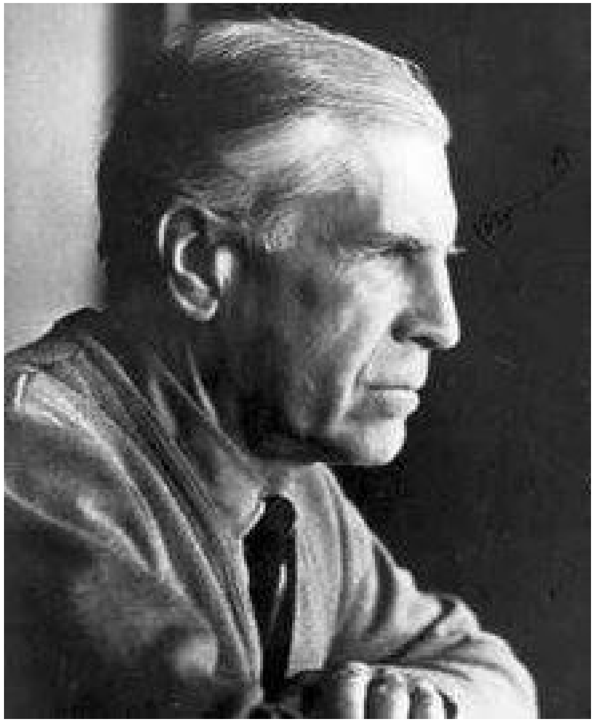

Figure 3: Lev S. Pontryagin (1908-1988)

- Core in stochastic control theory
- We will only cover the maximum principle in the deterministic case

Dynamic analysis in continuous time

To understand what PMP does, let's consider the following deterministic control problem:

$$\max_{c(\cdot)} \int_0^T e^{-\rho t} u(t, c(t), x(t)) \mathrm{d}t + f(x(T))$$

$$\text{s.t.} \quad \mathrm{d}x(t) = \mu(t, x(t), c(t)) \mathrm{d}t$$

Define the *(generalized) Hamiltonian*:

$$\mathcal{H}\left(c(t); t, x(t); \{\lambda_i(t)\}_{i=1,\dots,N}\right) := e^{-\rho t} u(t, c(t), x(t)) + \sum_{i=1}^{N} \lambda_i(t) \cdot \mu(t, x(t), c(t))$$

where

- $\lambda_i(t) \in \mathbb{R}$ is the *adjoint variable* or *co-state*, which is the shadow value of the state variable $x_i(t)$

Wait a minute! Does the Hamiltonian look familiar?

Dynamic analysis in continuous time

Recall the objective function of solving the policy functions in HJB:

$$\mathcal{H}\left(c; k, z; \nabla_{k,z} v; \nabla^2_{k,z} v\right) := u(t, c, x) + \mu(t, x, c)^T \cdot \nabla_x v + \frac{1}{2} \operatorname{tr}(\sigma(t, x, c)\sigma(t, x, c)^T \nabla^2_x v)$$

Drop the uncertainty and write each partial derivative:

$$\mathcal{H}\left(c; k, z; \{\partial v / \partial x_i\}_{i=1,\dots,N}\right) := u(t, c, x) + \sum_{i=1}^{N} \frac{\partial v}{\partial x_i} \cdot \mu_i(t, x, c)$$

Then, discount to $t = 0$:

$$\mathcal{H}\left(c; k, z; \{\partial v / \partial x_i\}_{i=1,\dots,N}\right) := e^{-\rho t} u(t, c, x) + \sum_{i=1}^{N} e^{-\rho t} \frac{\partial v}{\partial x_i} \cdot \mu_i(t, x, c)$$

cp. the Hamiltonian in the PMP:

$$\mathcal{H}\left(c(t); t, x(t); \{\lambda_i(t)\}_{i=1,\dots,N}\right) := e^{-\rho t} u(t, c(t), x(t)) + \sum_{i=1}^{N} \lambda_i(t) \cdot \mu(t, x(t), c(t))$$

They share the same form!

Dynamic analysis in continuous time

Intuitively,

$$\lambda_i(t) = e^{-\rho t} \frac{\partial v}{\partial x_i}$$

- maximizing the Hamiltonian $\Longrightarrow$ solving the policy functions
- dynamic of the adjoint variable $\Longleftrightarrow$ dynamics of the shadow value

In addition to finding policy functions, PMP also provides the dynamics of the adjoint variables, which gives the Euler equations when combined with the FOCs of the Hamiltonian

Dynamic analysis in continuous time

When optimal, the *adjoint equation* as a backward differential equation:

$$d\boldsymbol{\lambda}(t) = -\nabla_x \mathcal{H}\,\mathrm{d}t$$

or respectively $\forall i = 1, ..., N$:

$$\mathrm{d}\lambda_i(t) = -\frac{\partial \mathcal{H}}{\partial x_i}\mathrm{d}t = -\left\{ \frac{\partial(e^{-\rho t} \cdot u)}{\partial x_i} + \sum_{j=1}^{N} \lambda_j(t) \cdot \frac{\partial \mu_j}{\partial x_i} \right\}\mathrm{d}t$$

with a terminal condition (finite horizon):

$$\lambda_i(T) = \frac{\partial f}{\partial x_i}(x^*(T))$$

or a transversality condition (infinite horizon):

$$\lim_{t\to\infty} e^{-\rho t}\lambda_i(t) = 0$$

Intuitively consistent with the envelope theorem

Dynamic analysis in continuous time

Example: neo-classical growth model

$$\max_{c(\cdot)} \int_0^\infty e^{-\rho t} \log(c(t)) \mathrm{d}t$$

$$\text{s.t.} \quad \mathrm{d}k(t) = -\delta k(t) + zk(t)^\alpha - c(t)$$

Define Hamiltonian:

$$\mathcal{H}(c(t); t, k(t); \lambda_k(t)) := e^{-\rho t} \log(c(t)) + \lambda_k(t) \cdot [-\delta k(t) + zk(t)^\alpha - c(t)]$$

Policy function by maximizing the Hamiltonian:

$$c(t) = \left(e^{\rho t} \lambda_k(t)\right)^{-1} = \left(\frac{\partial v}{\partial k}\right)^{-1}$$

Adjoint equation:

$$\mathrm{d}\lambda_k(t) = -\lambda_k(t) \cdot [-\delta + \alpha z k(t)^{\alpha - 1}] \mathrm{d}t$$

Dynamic analysis in continuous time

Adjoint equation:

$$\mathrm{d}\lambda_k(t) = -\lambda_k(t) \cdot \left[-\delta + \alpha z k(t)^{\alpha-1}\right]\mathrm{d}t$$
$$\Longrightarrow \frac{\mathrm{d}\lambda_k(t)}{\lambda_k(t)} = \left[\alpha z k(t)^{\alpha-1} - \delta\right]\mathrm{d}t$$

- Interpretation: the *growth rate* of the shadow value of capital is equal to the *net marginal return* of increasing one unit of capital
- Trick: When $\frac{\mathrm{d}x(t)}{x(t)}$ is small, $\mathrm{d}\log(x(t))$ is a good approximation
  - Apply Ito's lemma if needed

$\Longrightarrow$ How to derive the Euler equation then?

Dynamic analysis in continuous time

Recall the FOC of the Hamiltonian:

$$\frac{1}{c(t)} = e^{\rho t}\lambda_k(t)$$

Knowing that $c > 0$, take the log:

$$-\log(c(t)) = \rho t + \log(\lambda_k(t))$$

Take time derivative:

$$-\frac{\mathrm{d}c(t)}{c(t)} = \rho \mathrm{d}t + \frac{\mathrm{d}\lambda_k(t)}{\lambda_k(t)}$$

Plugging it into the adjoint equation:

$$\frac{\mathrm{d}\lambda_k(t)}{\lambda_k(t)} = -\frac{\mathrm{d}c(t)}{c(t)} - \rho \mathrm{d}t = [\alpha z k(t)^{\alpha-1} - \delta]\mathrm{d}t$$

Dynamic analysis in continuous time

Plugging it into the adjoint equation:

$$\frac{\mathrm{d}\lambda_k(t)}{\lambda_k(t)} = -\frac{\mathrm{d}c(t)}{c(t)} - \rho \mathrm{d}t = [\alpha z k(t)^{\alpha-1} - \delta]\mathrm{d}t$$

Then, we get the Euler equation:

$$\frac{\mathrm{d}c(t)}{c(t)} = [\rho - (\alpha z k(t)^{\alpha-1} - \delta)]\mathrm{d}t$$

Interpretation:

- The Euler equation tells us how the consumption growth rate changes over time
- The consumption growth rate trade-off between the marginal return of capital and the subjective discount rate
- If it is more profitable to invest in capital: reduce consumption
- If it is less profitable to invest in capital: increase consumption

Dynamic analysis in continuous time

- *(Deterministic) steady state* is the long-run equilibrium of the model
  - No uncertainty (turned off)
  - No growth
  - No allocation change over time

In SS, nothing changes over time:

- $x(t) \equiv \overline{x} \Longrightarrow \mathrm{d}x(t) = 0$
- $c(t) \equiv \overline{c} \Longrightarrow \mathrm{d}c(t) = 0$

while

- the adjoint variables decays at the discounting rate: $\frac{\mathrm{d}\lambda(t)}{\lambda(t)} = -\rho \mathrm{d}t$
- or equivalently, $\mathrm{d}\frac{\partial v}{\partial x} = 0$

Dynamic analysis in continuous time

Consider the neo-classical growth model example:

$$\mathrm{d}k(t) = [-\delta k(t) + zk(t)^{\alpha} - c(t)]\mathrm{d}t$$
$$\frac{\mathrm{d}c(t)}{c(t)} = [\rho - (\alpha z k(t)^{\alpha-1} - \delta)]\mathrm{d}t$$

In SS,

$$\mathrm{d}k(t) = 0 \Longrightarrow -\delta\overline{k} + z\overline{k}^{\alpha} - \overline{c} = 0$$
$$\frac{\mathrm{d}c(t)}{c(t)} = 0 \Longrightarrow \rho - \left(\alpha z\overline{k}^{\alpha-1} - \delta\right) = 0$$

which gives us the steady state of the model:

$$\overline{k} = \left(\frac{\alpha z}{\rho + \delta}\right)^{\frac{1}{1-\alpha}}$$
$$\overline{c} = -\delta\overline{k} + z\overline{k}^{\alpha}$$

Compare with the discrete time version:

$$v(k) = \max_{c,k'} \log(c) + \beta v(k')$$

$$\text{s.t.} \quad c + k' = (1-\delta)k + zk^\alpha$$

Euler equation:

$$\frac{c'}{c} = \beta\left[1 - \delta + \alpha z (k')^{\alpha-1}\right]$$

In SS,

$$\bar{c} + \bar{k} = (1-\delta)\bar{k} + z\bar{k}^\alpha$$

$$\frac{\bar{c}}{\bar{c}} = 1 = \beta\left[1 - \delta + \alpha z \left(\bar{k}\right)^{\alpha-1}\right]$$

Steady state:

$$\bar{k} = \left(\frac{\alpha z}{\beta^{-1} - (1-\delta)}\right)^{\frac{1}{1-\alpha}}$$

$$\bar{c} = -\delta\bar{k} + z\bar{k}^\alpha$$

Dynamic analysis in continuous time

Bridging the two:

Discrete time version:

$$\bar{k} = \left(\frac{\alpha z}{\beta^{-1} - 1 + \delta} \right)^{\frac{1}{1-\alpha}}$$

Continuous time version:

$$\bar{k} = \left(\frac{\alpha z}{\rho + \delta} \right)^{\frac{1}{1-\alpha}}$$

By definition,

$$\beta = e^{-\rho \Delta t}$$

When ρ is small, the following approximation holds:

$$\beta^{-1} - 1 \approx \rho \Delta t$$

Let Δt be 1, the two versions are consistent with each other

Dynamic analysis in continuous time

Practice 1: solve the necessary condition between ρ and r of the following household problem for SS existing:

$$\rho v(a) = \max_{c} \frac{c^{1-\gamma}}{1-\gamma} + \frac{\partial v}{\partial a} \cdot (ra - c)$$

Hint:

- Step 1: define Hamiltonian
- Step 2: derive the policy function
- Step 3: derive the adjoint equation
- Step 4: derive the Euler equation
- Step 5: solve the SS of the model

Dynamic analysis in continuous time

Define Hamiltonian:

$$\mathcal{H} := e^{-\rho t}\frac{c^{1-\gamma}}{1-\gamma} + \lambda_a \cdot (ra - c)$$

Solve policy function:

$$c = \left(e^{\rho t}\lambda_a\right)^{-\frac{1}{\gamma}}$$

Adjoint equation:

$$\frac{\mathrm{d}\lambda_a}{\lambda_a} = -r\mathrm{d}t$$

Dynamic analysis in continuous time

Derive Euler equation:

$$\log(c) = -\frac{1}{\gamma}[\rho t + \log(\lambda_a)] \Longrightarrow \frac{\mathrm{d}\lambda_a}{\lambda_a} = -\gamma\frac{\mathrm{d}c}{c} - \rho \mathrm{d}t$$

$$\Longrightarrow \frac{\mathrm{d}c}{c} = \frac{1}{\gamma}(r - \rho)\mathrm{d}t$$

If SS exists, then c is constant over time such that $\frac{\mathrm{d}c}{c} = 0$

$$\frac{1}{\gamma}(r - \rho)\mathrm{d}t = 0$$

$$\Longrightarrow r = \rho$$

which is consistent with the intuition of the model, also the standard theory result in the discrete time model

Dynamic analysis in continuous time

Practice 2 (optional): Consider an entrepreneur who can invest in two assets: a liquid capital k of price 1 and an illiquid productive housing wealth h of price p.

$$\rho v(k,h) = \max_{c, I_h} \frac{c^{1-\gamma}}{1-\gamma}$$

$$+\frac{\partial v}{\partial k} \cdot \left[z k^{\alpha} h^{1-\alpha} - \delta_k k - c - \left(I_h + \frac{\psi}{2} I_h^2 \right) p \cdot h \right] + \frac{\partial v}{\partial h} \cdot (I_h - \delta_h) h$$

Question: Solve the house price p that allows the SS to hold

Hint:

- Step 1: define Hamiltonian, solve the policy functions
- Step 2: derive the adjoint equations (and the Euler equations if needed)
- Step 3: combining the above equations to solve the steady state conditions

Dynamic analysis in continuous time

Define Hamiltonian (discounting to $t = 0$):

$$\mathcal{H} := e^{-\rho t}\frac{c^{1-\gamma}}{1-\gamma} + \lambda_k \cdot \left[zk^\alpha h^{1-\alpha} - \delta_k k - c - \left(I_h + \frac{\psi}{2}I_h^2\right)p \cdot h\right] + \lambda_h \cdot (I_h - \delta_h)h$$

Policy functions:

$$c = (e^{\rho t}\lambda_k)^{-\frac{1}{\gamma}}, I_h = \frac{1}{\psi}\left[\frac{\lambda_h}{\lambda_k}\frac{1}{p} - 1\right]$$

Adjoint equations:

$$\frac{\mathrm{d}\lambda_k}{\lambda_k} = -[\alpha z k^{\alpha-1}h^{1-\alpha} - \delta_k]\mathrm{d}t$$

$$\frac{\mathrm{d}\lambda_h}{\lambda_h} = -\frac{\lambda_k}{\lambda_h} \cdot \left[(1-\alpha)zk^\alpha h^{-\alpha} - \left(I_h + \frac{\psi}{2}I_h^2\right)p\right]\mathrm{d}t - (I_h - \delta_h)\mathrm{d}t$$

In SS, the state equations

$$\mathrm{d}h = 0 \Longrightarrow \overline{I}_h = \delta_h$$

$$\mathrm{d}k = 0 \Longrightarrow z\overline{k}^{\alpha}\overline{h}^{1-\alpha} - \delta_k\overline{k} - \overline{c} - \left(\delta_h + \frac{\psi}{2}\delta_h^2\right)p \cdot \overline{h} = 0$$

the adjoint variables decays at the discounting rate ρ:

$$-\rho \mathrm{d}t = -\left[\alpha z\overline{k}^{\alpha-1}\overline{h}^{1-\alpha} - \delta_k\right]\mathrm{d}t$$

$$-\rho \mathrm{d}t = -\frac{\overline{\lambda_k}}{\overline{\lambda_h}} \cdot \left[(1-\alpha)z\overline{k}^{\alpha}\overline{h}^{-\alpha} - \left(\delta_h + \frac{\psi}{2}\delta_h^2\right)p\right]$$

Notice the SS also satisfies the policy functions:

$$\overline{I}_h = \delta_h = \frac{1}{\psi}\left[\frac{\overline{\lambda_h}}{\overline{\lambda_k}}\frac{1}{p} - 1\right] \Longrightarrow \frac{\overline{\lambda_h}}{\overline{\lambda_k}} = p(\psi\delta_h + 1)$$

Plugging it into the SS adjoint equation:

$$-\rho = -\alpha z \overline{k}^{\alpha-1} \overline{h}^{1-\alpha} + \delta_k$$

$$-\rho = -p(\psi\delta_h + 1) \cdot \left[(1-\alpha) z \overline{k}^{\alpha} \overline{h}^{-\alpha} - \left(\delta_h + \frac{\psi}{2}\delta_h^2 \right) p \right]$$

Cancel out k/h and get:

$$\left(\delta_h + \frac{\psi}{2}\delta_h^2 \right) p + (1-\alpha) z \left(\frac{\alpha z}{\delta_k + \rho} \right)^{\frac{\alpha}{1-\alpha}} = \frac{\rho}{p(\psi\delta_h + 1)}$$

This equation has two real roots, crazy but *analytical*

FYI: does this model have unique SS? If yes, how to prove it? If no, why?

FYI: can you derive the counterpart in discrete time?

(Optional) Numerical methods

- Numerical solutions are necessary for more complex continuous time models
- Nature of the problem: a free-boundary value PDE problem (fBVP)
- Available methods:
  - Finite difference methods (FDM)
  - Finite element methods (FEM)
  - Finite volume methods (FVM)
  - Spectral methods
  - Machine learning methods (e.g. neural networks)
  - Dynamic programming methods
  - ...
- Currently, FDM is the most common method in economics
  - Pros: straightforward, non-parametric
  - Cons: dimensionality curse, convergence, regular grid, ...

(Optional) Numerical methods

Finite difference method (FDM)

$$\rho v(t,x) = \max_c u(t,c,x) + \frac{\partial v}{\partial t} + \mu(t,x,c)^T \cdot \nabla_x v + \frac{1}{2}\operatorname{tr}(\sigma(t,x,c)\sigma(t,x,c)^T \nabla_x^2 v)$$

- This is a PDE:
  - $\frac{\partial v}{\partial t}$
  - $\nabla_x v = \left(\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_N}\right)^T$
  - $\nabla_x^2 v = \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)_{\{i,j=1\}}$
- Idea: use finite difference to approximate the derivatives
  - Forward difference, e.g. $\frac{\partial v}{\partial t} \approx \frac{v(t+\Delta t)-v(t)}{\Delta}t$
  - Backward difference, e.g. $\frac{\partial v}{\partial t} \approx \frac{v(t)-v(t-\Delta t)}{\Delta}t$
  - Central difference, e.g. $\frac{\partial v}{\partial t} \approx \frac{v(t+\Delta t)-v(t-\Delta t)}{2\Delta t}$
- For the 2nd-order derivatives, e.g. $\frac{\partial^2 v}{\partial x_i^2} \approx \frac{v(x+\Delta x_i)-2v(x)+v(x-\Delta x_i)}{\Delta x_i^2}$
- Which kind of appoximations to use is not trivial

(Optional) Numerical methods

- Practical FDM needs another single course
- Some concepts that you need to know:
  - Boundary conditions, free boundary problems, and singularities
  - Numerical scheme for FDM
  - FDM convergence analysis
    - Convergence
    - Consistency
    - Stability
    - Monotonicity
  - Barles-Souganidis theorem

Barles, G., & Souganidis, P. E. (1991). Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Analysis*, *4*(3), 271–283.

(Optional) Numerical methods

FYI,

- Lecture notes and code:
  - Benjamin Moll's website: [https://benjaminmoll.com/]
  - esp. applications in heterogeneous agent models and HANK
- Papers:

Kaplan, G., Moll, B., & Violante, G. L. (2018). Monetary policy according to HANK. *American Economic Review*, *108*(3), 697–743.

Frontier topics

- Heterogenous agent (HA) models e.g. HANK
- Perturbation methods for continuous time models, esp. HA models
- High-dimensional methods for continuous time models
- Machine learning methods for continuous time models

Bibliography

Barles, G., & Souganidis, P. E. (1991). Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Analysis*, *4*(3), 271–283.

Brunnermeier, M. K., & Sannikov, Y. (2014). A macroeconomic model with a financial sector. *American Economic Review*, *104*(2), 379–421.

Glawion, R. (2023). Sequence-Space Jacobians in Continuous Time. *Available at SSRN 4504829*.

Kaplan, G., Moll, B., & Violante, G. L. (2018). Monetary policy according to HANK. *American Economic Review*, *108*(3), 697–743.

Stokey, N. L. (2008). *The Economics of Inaction: Stochastic Control models with fixed costs*. Princeton University Press.

Yong, J., & Zhou, X. Y. (2012). *Stochastic controls: Hamiltonian systems and HJB equations* (Vol. 43). Springer Science & Business Media.