Session: Introduction to Continuous-Time Macroeconomic Modelling

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Macroeconomics II

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Plan

Core material

- Why continuous time?
- Review: stochastic calculus
- From discrete time to continuous time
  1. First introduce a time step Δt
  2. Scale payoffs, discounting, drifts, and shocks correctly
  3. Then take the limit $\Delta t \to 0$
- Dynamic programming and the Hamilton-Jacobi-Bellman (HJB) equation
- Non-stationary vs. stationary HJB conventions
- Deterministic steady state analysis

Optional / preview material

- Maximum principle as a tool for deriving Euler equations
- Introduction to numerical methods

Why continuous time?

- Many macroeconomic variables are naturally modeled as flows and stocks in continuous time
- Many advantages
  - Analytical tractability
  - Natural treatment of flow constraints, Poisson shocks, and Brownian shocks
  - Useful for HJB/KFE systems in heterogeneous-agent models
  - Some complex discrete-time models become cleaner in continuous time
- Standard in finance and many fields e.g. Merton's portfolio problem
- "Renaissance"
  - More departments are teaching continuous time in graduate even undergraduate courses
  - Distributional economics by Moll, Kaplan, and others
  - Heterogeneous Agent New Keynesian (HANK) models

What mathematical tools are involved?

Looks good, but the math is a bit scary!

- Stochastic calculus
  - Brownian motion
  - Ito's lemma
  - Stochastic differential equations (SDEs)
  - ...
- Stochastic control
  - Maximum principle (optional tool for deriving Euler equations)
  - Hamilton-Jacobi-Bellman (HJB) equation
  - Dynamic programming
  - ...
- Partial differential equations (PDE) and dynamic systems
  - 2nd order non-linear and pseudo-linear PDEs
  - Kolmogorov equations
  - ...
- Numerical PDE
  - Finite difference methods
  - Barles & Souganidis (1991) monotonic scheme analysis
  - ...

But don't worry, only a few of them are needed in this introductory session.

Reading list & expectations

Textbook & notes:

1. Benjamin Moll's notes: https://benjaminmoll.com/lectures/
2. Stokey, N. L. (2008). *The Economics of Inaction: Stochastic Control models with fixed costs*. Princeton University Press.

Suggested readings:

1. Yong, J., & Zhou, X. Y. (2012). *Stochastic controls: Hamiltonian systems and HJB equations* (Vol. 43). Springer Science & Business Media.
2. Kaplan, G., Moll, B., & Violante, G. L. (2018). Monetary policy according to HANK. *American Economic Review*, *108*(3), 697–743.
3. Brunnermeier, M. K., & Sannikov, Y. (2014). A macroeconomic model with a financial sector. *American Economic Review*, *104*(2), 379–421.
4. Glawion, R. (2023). Sequence-Space Jacobians in Continuous Time. *Available at SSRN 4504829*.

Reading list & expectations

After this session, you should be able to:

1. Understand core concepts and math formulations of continuous time economics
2. Understand standard economic models in continuous time and connect them to discrete time counterparts
3. Define economic problems in continuous time as stochastic control problems and HJB equations
4. Analytically solve the policy functions given an HJB equation
5. Analytically solve steady states (long-run equilibrium) of simple deterministic models
6. Learn where to find more information about continuous time economics

Review of stochastic calculus basics

Definition: A continuous time process $\{W(t)\}_{t\geq 0}$ taking values in $\mathbb{R}$ is called a *standard Brownian motion* (or *standard Wiener process*) if

1. $W(0) = 0$
2. $W(t) - W(s)$ is independent of the past history $\{W(r)\}_{r\leq s}$ for $0 \leq s < t$
3. $W(t) - W(s) \sim \mathcal{N}(0, t-s)$ for $0 \leq s < t$
4. $W(t)$ is continuous in t w.p.1

Multivariate Brownian motion: $W(t) = (W_1(t), W_2(t), ..., W_n(t))$ is a vector of n Brownian motions (possibly correlated)

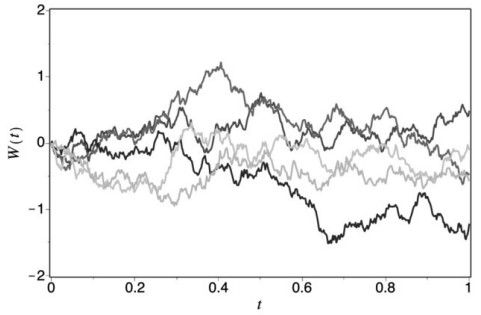


Figure 1: Sample paths of a standard Brownian motion[1]

[1]Source: Financial Mathematics: A Comprehensive Treatment in Continuous Time Volume II

Review of stochastic calculus basics

An n-dimensional stochastic process $X(t)$ is called a *diffusion process* if there exist functions $\mu(t,x)$ and $\sigma(t,x)$ such that

$$\mathrm{d}\ X(t) = \mu(t, X(t))\ \ \mathrm{d}\ t + \sigma(t, X(t))\ \ \mathrm{d}\ W(t)$$

where

- $W(t)$ is an m-dimensional Brownian motion
- $\mu(t, X(t)) \mapsto \mathbb{R}^n$ is a n-dimensional drift coefficient
- $\sigma(t, X(t)) \mapsto \mathbb{R}^{n\times m}$ is the diffusion/volatility coefficient

The equation is called a *stochastic differential equation* (SDE)

Properties:

- (Markovian) A diffusion process is a Markov process
- (Continuity) A diffusion process is continuous in t w.p.1

Review of stochastic calculus basics

Some commonly used processes:

- Ornstein-Uhlenbeck (OU) process: $dX(t) = \theta(\mu - X(t))\ \mathrm{d}\ t + \sigma\ \mathrm{d}\ W(t)$
  - Useful for modeling mean-reverting processes
  - Counterpart of AR(1) process: $x_{t+1} = \rho_D x_t + (1-\rho_D)\overline{x} + \varepsilon_{t+1}$ with $\rho_D = \exp(-\theta\Delta t)$
  - Solution: $X(t) = \mu + (X(0) - \mu)\exp(-\theta t) + \sigma\int_0^t \exp(-\theta(t-s))\ \mathrm{d}\ W(s)$
- Geometric Brownian motion (GBM): $\frac{dX(t)}{X(t)} = \mu\ \mathrm{d}\ t + \sigma\ \mathrm{d}\ W(t)$
  - Useful for modeling asset prices
  - Random walk with drift, non-negative
  - Solution: $X(t) = X(0)\exp\Big(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W(t)\Big)$
- Cox-Ingersoll-Ross (CIR) process: $dX(t) = \theta(\mu - X(t))\ \mathrm{d}\ t + \sigma\sqrt{X(t)}\ \mathrm{d}\ W(t)$
  - Useful for modeling interest rates
  - Non-negative, mean-reverting
- (Extra) Poisson process aka continuous-time Markov chain
  - Useful for modeling discrete state processes (e.g. unemployment)

Review of stochastic calculus basics

- **Ito's lemma** is the foundation of stochastic calculus
- Stochastic version of the chain rule in calculus
- Allows us to compute the infinitesimal change in a function of a stochastic process
- Used to derive the dynamics of a function of a diffusion process



Kiyoshi Ito (伊藤 清)
1915-2008
1970 at Cornell University

Review of stochastic calculus basics

Let $X(t)$ be a scalar diffusion process

$$dX(t) = \mu(t, X(t)) \text{ d } t + \sigma(t, X(t)) \text{ d } W(t)$$

and $f(t, x)$ be a twice continuously differentiable function. Then $f(t, X(t))$ is also a diffusion process and satisfies

$$\text{d } f = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} \right) \text{d } t + \sigma \frac{\partial f}{\partial x} \text{ d } W(t)$$

Useful rules:

- $\text{d } t \cdot \text{d } t = 0$, $\text{d } t \cdot \text{d } W(t) = 0$, and $\text{d } W(t) \cdot \text{d } W(t) = \text{d } t$
- Addition rule: $\text{d } (X \pm Y) = \text{d } X \pm \text{d } Y$
- Product rule: $\text{d } (XY) = X \text{ d } Y + Y \text{ d } X + \text{d } X \text{ d } Y$
- For ratios or other nonlinear transformations, apply Ito's lemma directly to the relevant function, e.g. $f(X, Y) = \frac{X}{Y}$

Review of stochastic calculus basics

Example

Let $p(t)$ be the process of house price, $h(t)$ be the process of household housing stock:

$$\frac{\mathrm{d}\ p(t)}{p(t)} = \mu_p\ \mathrm{d}\ t + \sigma_p\ \mathrm{d}\ W(t)$$

$$\mathrm{d}\ h(t) = I\ \mathrm{d}\ t$$

What is the SDE of the housing value $V(t) \coloneqq p(t)h(t)$?

$$\begin{aligned}
&\mathrm{d}\ V(t) = p(t)\ \mathrm{d}\ h(t) + h(t)\ \mathrm{d}\ p(t) + \mathrm{d}\ p(t)\ \mathrm{d}\ h(t) \\
&= p(t) \cdot [I\ \mathrm{d}\ t] + h(t) \cdot \left[\mu_p p(t)\ \mathrm{d}\ t + \sigma_p p(t)\ \mathrm{d}\ W(t)\right] + [I\ \mathrm{d}\ t] \cdot \left[\mu_p p(t)\ \mathrm{d}\ t + \sigma_p p(t)\ \mathrm{d}\ W(t)\right] \\
&= \left\{p(t) \cdot I\ \mathrm{d}\ t + \mu_p h(t)p(t)\ \mathrm{d}\ t + h(t)\sigma_p p(t)\ \mathrm{d}\ W(t)\right\} + \left\{I\mu_p p(t)\ \mathrm{d}\ t \cdot \mathrm{d}\ t + I\sigma_p p(t)\ \mathrm{d}\ t \cdot \mathrm{d}\ W(t)\right\} \\
&= \left[p(t) \cdot I + \mu_p V(t)\right]\ \mathrm{d}\ t + \sigma_p V(t)\ \mathrm{d}\ W(t) + \{0 + 0\}
\end{aligned}$$

Review of stochastic calculus basics

Multivariate Ito's lemma:

Let

$$d\boldsymbol{X}(t) = \boldsymbol{\mu}(t, \boldsymbol{X}(t))\ \mathrm{d}\ t + \boldsymbol{\sigma}(t, \boldsymbol{X}(t))\ \mathrm{d}\ \boldsymbol{W}(t)$$
$$\boldsymbol{X}(t) \in \mathbb{R}^n, \boldsymbol{W}(t) \in \mathbb{R}^m, \boldsymbol{\mu} \mapsto \mathbb{R}^n, \boldsymbol{\sigma} \mapsto \mathbb{R}^{n \times m}$$

and $f(t, \boldsymbol{x}) \mapsto \mathbb{R}$ is a twice continuously differentiable function. Then

$$\mathrm{d}\ f(t, \boldsymbol{X}(t)) = \left\{ \frac{\partial f}{\partial t} + \boldsymbol{\mu}^T \nabla_X f + \frac{1}{2} \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{\sigma}^T \nabla_X^2 f) \right\} \mathrm{d}\ t + (\nabla_X f)^T \boldsymbol{\sigma}\ \mathrm{d}\ \boldsymbol{W}(t)$$

where

- $\nabla_X f \in \mathbb{R}^n$ is the gradient of f with respect to X
- $\nabla_X^2 f \in \mathbb{R}^{n \times n}$ is the Hessian of f with respect to X

- Useful in models with multiple uncertainties. FYI, no need to memorize

From discrete time to continuous time

In macroeconomic models, we usually care lifetime problems like:

$$v_0(x_0) = \max_{\{c_t\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{T-1} \beta^t u(t, c_t, x_t) + \beta^T f(x_T) \right\}$$

$$\text{s.t.} \quad x_{t+1} = g(t, x_t, c_t)$$

$$c_t \in \mathbb{C}_t, x_t \in \mathbb{X}_t$$

- x_0 is the initial state
- x is the state vector (e.g. capital, asset)
- c is the control vector (e.g. consumption, investment)
- $u(t, c, x)$ is the instantaneous utility function
- $f(x)$ is the terminal utility function
- $g(t, x, c)$ is the state equation or law of motion (e.g. budget constraint)
- $\mathbb{C}_t$ is the set of admissible controls, and $\mathbb{X}_t$ is the set of admissible states (state constraints)

From discrete time to continuous time

Example: Neo-classical growth model

$$v_0(k_0) = \max_{\{c_t\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \log(c_t) \right\}$$
$$\text{s.t.} \quad \forall t, k_{t+1} = (1-\delta)k_t + k_t^\alpha - c_t$$
$$c_t \geq 0, k_t \geq 0$$

where

- planning horizon: $T = \infty$
- state variables: k_t (capital)
- control variables: c_t (consumption)
- instantaneous utility: $\log(c)$
- no terminal utility in the infinite-horizon formulation
- transversality condition: the shadow value of capital times capital must vanish asymptotically
- Admissible sets
  - $\mathbb{C}_t = \{c_t \geq 0\}$
  - $\mathbb{X}_t = \{k_t \geq 0\}$

From discrete time to continuous time

By Bellman's principle, we can rewrite the problem as the following *dynamic programming* (DP) problem (aka *Bellman equation*):

$$v(t,x) = \max_{c_t} \mathbb{E}_t\{u(t,c_t,x_t) + \beta v(t+1, x_{t+1})\}$$
$$\text{s.t.} \quad x_{t+1} = g(t,x_t,c_t); c_t \in \mathbb{C}_t, x_t \in \mathbb{X}_t$$

If the problem is *time homogeneous*[1], we can drop the time index:

$$v(x) = \max_{c} \mathbb{E}\{u(c,x) + \beta v(x')\}$$
$$\text{s.t.} \quad x' = g(x,c); c \in \mathbb{C}, x \in \mathbb{X}$$

Example: Neo-classical growth model

$$v(k) = \max_{c} \mathbb{E}\{\log(c) + \beta v(k')\}$$
$$\text{s.t.} \quad k' = (1-\delta)k + k^\alpha - c$$
$$c \geq 0, k \geq 0$$

[1] i.e. conditions do not depend on t or the history of the states.

From discrete time to continuous time

Question: discrete time $\longrightarrow$ continuous time, how?

$$v_0(x_0) = \max_{\{c_t\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{T-1} \beta^t u(t, c_t, x_t) + \beta^T f(x_T) \right\} \to ?$$

$$\text{s.t.} \quad x_{t+1} = g(t, x_t, c_t) \qquad \to ?$$

$$c_t \in \mathbb{C}_t, x_t \in \mathbb{X}_t \qquad \to ?$$

A safer way is not to replace x_{t+1} mechanically by $\mathrm{d}x(t)$.

From discrete time to continuous time

First introduce a time step $\Delta t > 0$ and a time grid $t_n = n\Delta t$.

A discrete-time approximation to a continuous-time problem should be written as:

$$v^{\Delta t}(0, x_0) = \max_{\{c_n\}_n} \mathbb{E}_0 \left\{ \sum_{n=0}^{N-1} e^{-\rho t_n} u(t_n, c_n, x_n) \Delta t + e^{-\rho T} f(x_N) \right\}$$

where $T = N\Delta t$.

Key point:

- $u(t, c, x)$ is a flow payoff per unit of time
- the payoff over one short period is therefore $u(t_n, c_n, x_n)\Delta t$
- $\beta_{\Delta t} = e^{-\rho \Delta t}$, hence $(\beta_{\Delta t})^n = e^{-\rho t_n}$

From discrete time to continuous time

The state equation must also be scaled before taking the limit:

$$x_{n+1} - x_n = \mu(t_n, x_n, c_n)\Delta t + \sigma(t_n, x_n, c_n)\sqrt{\Delta t}\varepsilon_{n+1}, \quad \varepsilon_{n+1} \sim N(0, I_m)$$

Then, as $\Delta t \to 0$:

$$\sum_n e^{-\rho t_n} u(t_n, c_n, x_n)\Delta t \to \int_0^T e^{-\rho t} u(t, c(t), x(t)) \, \mathrm{d}t$$

$$x_{n+1} - x_n \to \mathrm{d}x(t) = \mu(t, x(t), c(t)) \, \mathrm{d}t + \sigma(t, x(t), c(t)) \, \mathrm{d}W(t)$$

Interpretation:

- drift terms are order Δt
- Brownian shocks are order $\sqrt{\Delta t}$
- continuous time is a limit of correctly scaled short-period problems

From discrete time to continuous time

Then generally, we can write the continuous time problem as the following *stochastic control problem*:

$$v_0(x_0) = \max_{c(\cdot)} \mathbb{E}_0 \left\{ \int_0^T e^{-\rho t} u(t, c(t), x(t)) \, \mathrm{d}\, t + e^{-\rho T} f(x(T)) \right\}$$

$$\text{s.t. } \mathrm{d}\, x(t) = \mu(t, x(t), c(t)) \, \mathrm{d}\, t + \sigma(t, x(t), c(t)) \, \mathrm{d}\, W(t)$$

$$c(t) \in \mathbb{C}_t, x(t) \in \mathbb{X}_t$$

where

- ρ is the continuous-time discount rate; on a grid with step Δt, the matching discrete discount factor is $\beta_{\Delta t} = e^{-\rho \Delta t}$
- $x(\cdot)$ is the *(controlled) state process*, $\forall t, x(t) \in \mathbb{R}^n$
- $c(\cdot)$ is the *control process*, $\forall t, c(t) \in \mathbb{R}^k$
- $W(\cdot)$ is the *Brownian motion*, $\forall t, W(t) \in \mathbb{R}^m$
- $\mu(t, x, c) \mapsto \mathbb{R}^n$ is the drift coefficient
- $\sigma(t, x, c) \mapsto \mathbb{R}^{n \times m}$ is the diffusion/volatility coefficient

From discrete time to continuous time

Example: Neo-classical growth model

Discrete time:

$$v_0(k_0) = \max_{\{c_t, k_{t+1}\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \log(c_t) \right\}$$

$$\text{s.t.} \quad \forall t, k_{t+1} = (1-\delta)k_t + k_t^\alpha - c_t$$

$$c_t \geq 0, k_t \geq 0$$

Continuous time:

$$v_0(k_0) = \max_{c(\cdot)} \mathbb{E}_0 \left\{ \int_0^{\infty} e^{-\rho t} \log(c(t)) \, \mathrm{d}\, t \right\}$$

$$\text{s.t.} \quad \mathrm{d}\, k(t) = [-\delta k(t) + k(t)^\alpha - c(t)] \, \mathrm{d}\, t$$

$$c(t) \geq 0, k(t) \geq 0$$

Notice:

- continuous discounting
- change of the control variables
- the state equation tells the *change* of $k(t)$ now
- a standard transversality condition is $\lim_{t\to\infty} \lambda_{k(t)} k(t) = 0$, where $\lambda_{k(t)}$ is the present-value shadow value of capital
- the u, c are interpreted as flow/rate rather than amount, cp. c_t in discrete time and $\int_t^{t+\Delta t} c(t) \, \mathrm{d}\, t$ in continuous time

From discrete time to continuous time

Practice 1: write the following household consumption-leisure problem in continuous time:

$$v_0(a_0) = \max_{\{c_t, a_{t+1}, n_t\}_t} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma}}{1-\gamma} - \frac{n_t^{1+\nu}}{1+\nu} \right]$$

$$\text{s.t.} \quad c_t + a_{t+1} = (1+r)a_t + wn_t$$

From discrete time to continuous time

Practice 1: write the following household consumption-leisure problem in continuous time:

$$v_0(a_0) = \max_{\{c_t, a_{t+1}, n_t\}_t} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma}}{1-\gamma} - \frac{n_t^{1+\nu}}{1+\nu} \right]$$
$$\text{s.t.} \quad c_t + a_{t+1} = (1+r)a_t + wn_t$$

Continuous time version:

$$v_0(a_0) = \max_{c(\cdot), n(\cdot)} \int_0^{\infty} e^{-\rho t} \left[\frac{c(t)^{1-\gamma}}{1-\gamma} - \frac{n(t)^{1+\nu}}{1+\nu} \right] \mathrm{d}\ t$$
$$\text{s.t.} \quad \mathrm{d}\ a(t) = \{ra(t) + wn(t) - c(t)\}\ \mathrm{d}\ t$$

From discrete time to continuous time

Practice 2: write the following entrepreneur's problem in continuous time:

$$v_0(k_0, z_0) = \max_{\{c_t, k_{t+1}, n_t\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \log(c_t) \right\}$$

$$\text{s.t.} \quad c_t + k_{t+1} = z_t k_t^{\alpha} n_t^{1-\alpha} - w n_t$$

$$z_t \sim \text{AR}(1) \text{ process}$$

Hint: use OU process $\mathrm{d}\, z(t) = \theta(\mu - z(t))\, \mathrm{d}\, t + \sigma\, \mathrm{d}\, W(t)$

From discrete time to continuous time

Practice 2: write the following entrepreneur's problem in continuous time:

$$v_0(k_0, z_0) = \max_{\{c_t, k_{t+1}, n_t\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \log(c_t) \right\}$$

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$$z_t \sim \text{AR}(1) \text{ process}$$

Hint: use OU process $\mathrm{d}\ z(t) = \theta(\mu - z(t))\ \mathrm{d}\ t + \sigma\ \mathrm{d}\ W(t)$

Continuous time version:

$$v_0(k_0, z_0) = \max_{c(\cdot), n(\cdot)} \mathbb{E}_0 \left\{ \int_0^{\infty} e^{-\rho t} \log(c(t))\ \mathrm{d}\ t \right\}$$

$$\text{s.t.} \quad \mathrm{d}\ k(t) = \left\{ z(t) k(t)^{\alpha} n(t)^{1-\alpha} - w n(t) - c(t) \right\} \mathrm{d}\ t$$

$$\mathrm{d}\ z(t) = \theta(\mu - z(t))\ \mathrm{d}\ t + \sigma\ \mathrm{d}\ W(t)$$

From discrete time to continuous time

Practice 3: write the following price setter's problem with Rotemberg price rigidity in continuous time:

$$v_0(p_0, y_0) = \max_{\{p_{t+1}\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t y_t \cdot \left[p_t - \frac{\varphi}{2} (p_{t+1} - p_t)^2 \right] \right\}$$

$$\text{s.t.} \quad y_t \sim \text{AR}(1) \text{ process}$$

Hint:

- define the instantaneous inflation rate $\pi(t) := \frac{\dot{p}(t)}{p(t)}$
- specify a continuous-time Rotemberg flow cost $\frac{\varphi}{2}\pi(t)^2$
- this is not a mechanical limit of $\frac{\varphi}{2}(p_{t+1} - p_t)^2$: a literal square is order $(\Delta t)^2$ and requires parameter rescaling
- use OU process $\mathrm{d}\, y(t) = \theta(\overline{y} - y(t))\, \mathrm{d}\, t + \sigma\, \mathrm{d}\, W(t)$

From discrete time to continuous time

Practice 3: write the following price setter's problem with Rotemberg price rigidity in continuous time:

$$v_0(p_0, y_0) = \max_{\{p_{t+1}\}_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t y_t \cdot \left[p_t - \frac{\varphi}{2} (p_{t+1} - p_t)^2 \right] \right\}$$

$$\text{s.t.} \quad y_t \sim \text{AR}(1) \text{ process}$$

Hint:

- define the instantaneous inflation rate $\pi(t) := \frac{\dot{p}(t)}{p(t)}$
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- this is not a mechanical limit of $\frac{\varphi}{2}(p_{t+1} - p_t)^2$: a literal square is order $(\Delta t)^2$ and requires parameter rescaling
- use OU process $\mathrm{d}\, y(t) = \theta(\overline{y} - y(t))\, \mathrm{d}\, t + \sigma\, \mathrm{d}\, W(t)$

$$v_0(p_0, y_0) = \max_{\pi(\cdot)} \mathbb{E}_0 \left\{ \int_0^{\infty} e^{-\rho t} p(t) y(t) \cdot \left[1 - \frac{\varphi}{2} \pi(t)^2 \right] \mathrm{d}\, t \right\}$$

$$\text{s.t.} \quad \mathrm{d}\, p(t) = \pi(t) p(t)\, \mathrm{d}\, t$$

$$\mathrm{d}\, y(t) = \theta(\overline{y} - y(t))\, \mathrm{d}\, t + \sigma\, \mathrm{d}\, W(t)$$

Dynamic programming in continuous time

- Bellman's principle applies to the continuous time case as well

$$v(t,x) = \max_{c_t} \mathbb{E}_t\{u(t,c_t,x_t) + \beta v(t+1, x_{t+1})\}$$
$$\text{s.t.} \quad x_{t+1} = g(t,x_t,c_t); c_t \in \mathbb{C}_t, x_t \in \mathbb{X}_t$$

$$\downarrow\downarrow\downarrow$$

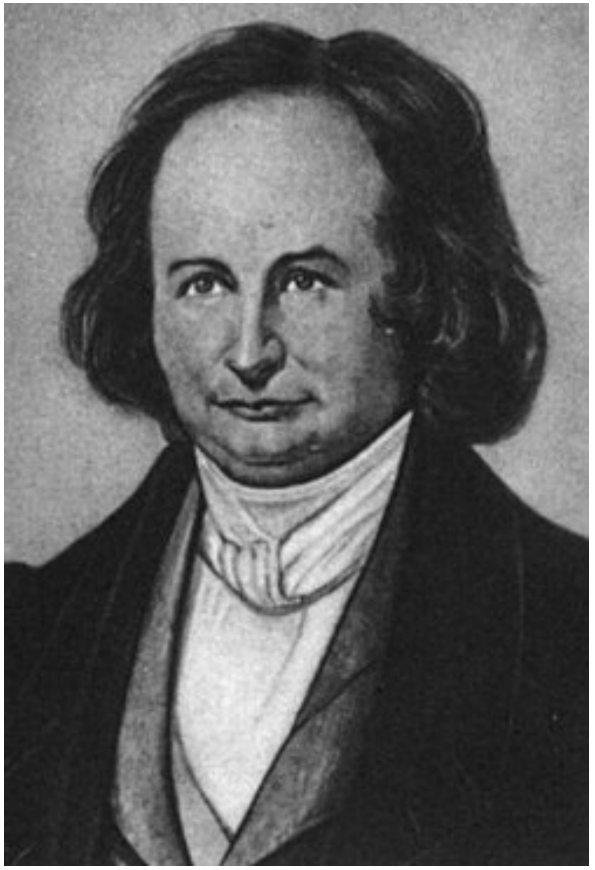
$$v(t,x) = \max_{c_{[t,t+\Delta t)}} \mathbb{E}_t \left\{ \int_t^{t+\Delta t} e^{-\rho(s-t)} u(s,c(s),x(s)) \, \mathrm{d}\, s + e^{-\rho \Delta t} v(t+\Delta t, x(t+\Delta t)) \right\}$$
$$\text{s.t.} \quad \mathrm{d}\, x(t) = \mu(t,x(t),c(t)) \, \mathrm{d}\, t + \sigma(t,x(t),c(t)) \, \mathrm{d}\, W(t)$$
$$c(t) \in \mathbb{C}_t, x(t) \in \mathbb{X}_t$$

- $\Delta t > 0$ is the time increment
- Relative discounting convention: from the viewpoint of time t, payoff at time s is discounted by $e^{-\rho(s-t)}$
- Can be solved exactly by the same method as in discrete time: value function iteration, policy function iteration, ...
- Smaller Δt, closer to the true solution. But what if $\Delta t \to 0$? what math tools are needed?

HJB equation



William Hamilton



Carl Jacobi



Richard Bellman

Hamilton-Jacobi-Bellman (HJB) equation

HJB equation

- By expanding the continuous time Bellman equation[1], we can derive the following *Hamilton-Jacobi-Bellman (HJB) equation*:

$$0 = \max_{c} \left\{ u(t,c,x) + \frac{\partial v}{\partial t} + \mu(t,x,c)^T \cdot \nabla_x v + \frac{1}{2} \operatorname{tr}\left(\sigma(t,x,c)\sigma(t,x,c)^T \nabla_x^2 v\right) - \rho v(t,x) \right\}$$

$$\text{s.t.} \quad c \in \mathbb{C}_t, x \in \mathbb{X}_t$$

By defining the *2nd-order infinitesimal generator*

$$\mathcal{L}^{c[v]} := \frac{\partial v}{\partial t} + \mu(t,x,c)^T \cdot \nabla_x v + \frac{1}{2} \operatorname{tr}\left(\sigma(t,x,c)\sigma(t,x,c)^T \nabla_x^2 v\right)$$

The HJB equation is sometimes written in the following *operator form*:

$$\max_{c \in \mathbb{C}_t} \left\{ u(t,c,x) + \mathcal{L}^{c[v]} - \rho v(t,x) \right\} = 0$$

[1] Chapter 4.3 of Yong & Zhou (2012), FYI

HJB equation

$$0 = \max_{c} \left\{ u(t,c,x) + \frac{\partial v}{\partial t} + \mu(t,x,c)^T \cdot \nabla_x v + \frac{1}{2} \operatorname{tr}\left(\sigma(t,x,c)\sigma(t,x,c)^T \nabla_x^2 v\right) - \rho v(t,x) \right\}$$

Tips

- $\mu(t,x,c)^T \cdot \nabla_x v \in \mathbb{R}$, inner product of the state drifts and the gradients of the value function
- $\sigma(t,x,c)$ can be non-square and need not be symmetric; but $\sigma(t,x,c)\sigma(t,x,c)^T$ is the covariance matrix and must be symmetric positive semidefinite
- $\frac{\partial v}{\partial t}$, the time derivative of the value function, ignored in time homogeneous problems where t can be ignored[1]
- The red term is the *drift term* (sometimes related to a flux term in PDE language), which tells the mean or deterministic part of how the value function changes over time
- the blue term is called the *diffusion term* which tells the *risk adjustment* of the value function due to the uncertainty

[1] $\mathrm{d}\, t = 1\, \mathrm{d}\, t$, thus $1 \cdot \frac{\partial v}{\partial t}$

HJB equation

$$0 = \max_{c} \left\{ u(t,c,x) + \frac{\partial v}{\partial t} + \mu(t,x,c)^T \cdot \nabla_x v + \frac{1}{2} \operatorname{tr}(\sigma(t,x,c)\sigma(t,x,c)^T \nabla_x^2 v) - \rho v(t,x) \right\}$$

Compared with the discrete time Bellman equation:

- No explicit expectation operator appears in the final HJB: the local conditional expectation over an infinitesimal time increment is summarized by the infinitesimal generator $\mathcal{L}^{c[v]}$
  - this is one reason why continuous-time dynamic programming is analytically convenient
- split of the risk adjustment
- HJB equation is a special kind of nonlinear partial differential equation (PDE) in t and x (state variables)
  - stationary second-order HJBs are often elliptic or degenerate elliptic (椭圆/退化椭圆)
  - time-dependent second-order HJBs are often parabolic-type (抛物型)
  - without diffusion terms, the HJB is first-order rather than second-order

Non-stationary vs. stationary conventions

Non-stationary / time-dependent problem

$$v(t,x) = \max_{c(\cdot)} \mathbb{E}_{t,x} \left\{ \int_t^T e^{-\rho(s-t)} u(s, c(s), x(s)) \, \mathrm{d}s + e^{-\rho(T-t)} f(x(T)) \right\}$$

The corresponding HJB is:

$$0 = \max_{c \in \mathbb{C}_t} \left\{ u(t,c,x) + \frac{\partial v}{\partial t} + \mu(t,x,c)^T \cdot \nabla_x v + \frac{1}{2} \operatorname{tr}\left(\sigma \sigma^T \nabla_x^2 v\right) - \rho v(t,x) \right\}$$

with terminal condition $v(T,x) = f(x)$.

Non-stationary vs. stationary conventions

Stationary / time-homogeneous infinite-horizon problem

If preferences, constraints, and laws of motion do not depend explicitly on calendar time, write:

$$v(x) = \max_{c(\cdot)} \mathbb{E}_x \left\{ \int_0^\infty e^{-\rho s} u(c(s), x(s)) \, \mathrm{d}s \right\}$$

The HJB becomes:

$$\rho v(x) = \max_{c \in \mathbb{C}} \left\{ u(c, x) + \mu(x, c)^T \cdot \nabla_x v + \frac{1}{2} \operatorname{tr}\left(\sigma(x, c)\sigma(x, c)^T \nabla_x^2 v\right) \right\}$$

No v_t term appears because the continuation problem is identical at every calendar date.

Examples: economic models as HJB equations

Example: Stochastic neo-classical growth model

$$v(0,k(0),z(0)) = \max_{c(\cdot)} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} \log(c(t)) \, \mathrm{d}\, t \right\}$$

$$\text{s.t.} \quad \mathrm{d}\, k(t) = [-\delta k(t) + z(t)k(t)^\alpha - c(t)] \, \mathrm{d}\, t$$

$$\mathrm{d}\, z(t) = \theta(\mu - z(t)) \, \mathrm{d}\, t + \sigma \, \mathrm{d}\, W(t)$$

$$\downarrow\downarrow\downarrow$$

$$v(t,k(t),z(t)) = \max_{c_{[t,t+\Delta t)}} \mathbb{E}_t \left\{ \int_t^{t+\Delta t} e^{-\rho(s-t)} \log(c(s)) \, \mathrm{d}\, s + e^{-\rho \Delta t} v(t+\Delta t, k(t+\Delta t), z(t+\Delta t)) \right\}$$

$$\text{s.t.} \quad \mathrm{d}\, k(t) = (-\delta k(t) + z(t)k(t)^\alpha - c(t)) \, \mathrm{d}\, t$$

$$\mathrm{d}\, z(t) = \theta(\mu - z(t)) \, \mathrm{d}\, t + \sigma \, \mathrm{d}\, W(t)$$

$$\downarrow\downarrow\downarrow$$

$$0 = \max_c \left\{ \log(c) + \frac{\partial v}{\partial t} + (-\delta k(t) + z(t)k(t)^\alpha - c(t)) \cdot \frac{\partial v}{\partial k} + \theta(\mu - z(t)) \cdot \frac{\partial v}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial z^2} - \rho v(t,k(t),z(t)) \right\}$$

$$\downarrow\downarrow\downarrow \text{ (time homogeneity)}$$

$$\rho v(k,z) = \max_c \log(c) + (-\delta k + z k^\alpha - c) \cdot \frac{\partial v}{\partial k} + \theta(\mu - z) \cdot \frac{\partial v}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial z^2}$$

Examples: economic models as HJB equations

Practice 1: write the HJB equation for the following household consumption-leisure problem in continuous time:

$$\max_{c(\cdot),n(\cdot)} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} \left[\frac{c(t)^{1-\gamma}}{1-\gamma} - \frac{n(t)^{1+\nu}}{1+\nu} \right] \mathrm{d}\, t \right\}$$

$$\text{s.t. } \mathrm{d}\, a(t) = \{ra(t) + w(t)n(t) - c(t)\}\, \mathrm{d}\, t$$

$$\mathrm{d}\, w(t) = \theta(\mu - w(t))\, \mathrm{d}\, t + \sigma\, \mathrm{d}\, W(t)$$

Hint: time homogeneous (does not depend on t or the history of the states)

Examples: economic models as HJB equations

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$$\mathrm{d}\, w(t) = \theta(\mu - w(t))\, \mathrm{d}\, t + \sigma\, \mathrm{d}\, W(t)$$

Hint: time homogeneous (does not depend on t or the history of the states)

HJB equation:

$$\rho v(a,w) = \max_{c,n} \frac{c^{1-\gamma}}{1-\gamma} - \frac{n^{1+\nu}}{1+\nu} + \frac{\partial v}{\partial a} \cdot (ra + wn - c) + \frac{\partial v}{\partial w} \cdot (\theta(\mu - w)) + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial w^2}$$

Examples: economic models as HJB equations

Practice 2: write the HJB equation for the following price setter's problem with Rotemberg price rigidity in continuous time:

$$v_0(p_0, y_0) = \max_{\pi(\cdot)} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} p(t) y(t) \cdot \left[1 - \frac{\varphi}{2} \pi(t)^2 \right] \mathrm{d}\, t \right\}$$

$$\text{s.t.} \quad \mathrm{d}\, p(t) = \pi(t) p(t)\, \mathrm{d}\, t$$

$$\mathrm{d}\, y(t) = \theta(\overline{y} - y(t))\, \mathrm{d}\, t + \sigma\, \mathrm{d}\, W(t)$$

Here $\frac{\varphi}{2}\pi(t)^2$ is specified directly as a continuous-time flow adjustment cost in the inflation rate. It should not be read as the literal limit of $\frac{\varphi}{2}(p_{t+1} - p_t)^2$ without a corresponding rescaling of φ.

Examples: economic models as HJB equations

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Here $\frac{\varphi}{2}\pi(t)^2$ is specified directly as a continuous-time flow adjustment cost in the inflation rate. It should not be read as the literal limit of $\frac{\varphi}{2}(p_{t+1} - p_t)^2$ without a corresponding rescaling of φ.

HJB equation:

$$\rho v(p, y) = \max_{\pi} py \cdot \left[1 - \frac{\varphi}{2} \pi^2 \right] + \frac{\partial v}{\partial p} \cdot \pi p + \frac{\partial v}{\partial y} \cdot \theta(\overline{y} - y) + \frac{1}{2} \sigma^2 \frac{\partial^2 v}{\partial y^2}$$

Examples: economic models as HJB equations

Practice 3: write the HJB equation for the following homeowner's problem:

$$v_0(a_0, h_0, z_0) = \max_{c(\cdot), I(\cdot), n(\cdot)} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} \left[\frac{\left[c(t)^\alpha h(t)^{1-\alpha}\right]^{1-\gamma}}{1-\gamma} - \frac{n(t)^{1+\nu}}{1+\nu} \right] \mathrm{d}\, t \right\}$$

$$\text{s.t. } \mathrm{d}\, a(t) = \left\{ r a(t) + z(t) n(t) - c(t) - \left[I(t) + \frac{\psi}{2} I(t)^2 \right] p \cdot h(t) \right\} \mathrm{d}\, t$$

$$\mathrm{d}\, h(t) = I(t) h(t)\, \mathrm{d}\, t$$

$$\mathrm{d}\, z(t) = \theta(\mu - z(t))\, \mathrm{d}\, t + \sigma\, \mathrm{d}\, W(t)$$

Examples: economic models as HJB equations

Practice 3: write the HJB equation for the following homeowner's problem:

$$v_0(a_0, h_0, z_0) = \max_{c(\cdot), I(\cdot), n(\cdot)} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} \left[\frac{\left[c(t)^\alpha h(t)^{1-\alpha}\right]^{1-\gamma}}{1-\gamma} - \frac{n(t)^{1+\nu}}{1+\nu} \right] \mathrm{d}\, t \right\}$$

$$\text{s.t. } \mathrm{d}\, a(t) = \left\{ r a(t) + z(t) n(t) - c(t) - \left[I(t) + \frac{\psi}{2} I(t)^2 \right] p \cdot h(t) \right\} \mathrm{d}\, t$$

$$\mathrm{d}\, h(t) = I(t) h(t)\, \mathrm{d}\, t$$

$$\mathrm{d}\, z(t) = \theta(\mu - z(t))\, \mathrm{d}\, t + \sigma\, \mathrm{d}\, W(t)$$

HJB equation:

$$\rho v(a, h, z) = \max_{c, I, n} \frac{\left[c^\alpha h^{1-\alpha}\right]^{1-\gamma}}{1-\gamma} - \frac{n^{1+\nu}}{1+\nu}$$

$$+ \frac{\partial v}{\partial a} \cdot \left(r a + z n - c - \left[I + \frac{\psi}{2} I^2 \right] p \cdot h \right) + \frac{\partial v}{\partial h} \cdot I h + \frac{\partial v}{\partial z} \cdot (\theta(\mu - z)) + \frac{1}{2} \sigma^2 \frac{\partial^2 v}{\partial z^2}$$

Examples: economic models as HJB equations

A more general example:

$$\rho v(x) = \max_c u(c) + \nabla_x v \cdot \mu + \frac{1}{2} \operatorname{tr}\left[\sigma \cdot \sigma^T \cdot \nabla_x^2 v\right]$$

$$\text{s.t. } \mathrm{d} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \mathrm{d}\, t + \underbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \end{bmatrix}}_{2\times 3} \mathrm{d} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}$$

Notice: the risks may be correlated

The flux term:

$$\nabla_x v \cdot \mu = \frac{\partial v}{\partial x_1} \mu_1 + \frac{\partial v}{\partial x_2} \mu_2$$

The diffusion term:

$$\frac{1}{2}\operatorname{tr}(\sigma\cdot\sigma^T\cdot\nabla_x^2 v)=\frac{1}{2}\operatorname{tr}\left\{\underbrace{\begin{bmatrix}\sigma_{11}&\sigma_{12}&\sigma_{13}\\\sigma_{21}&\sigma_{22}&\sigma_{23}\end{bmatrix}}_{2\times3}\cdot\underbrace{\begin{bmatrix}\sigma_{11}&\sigma_{21}\\\sigma_{12}&\sigma_{22}\\\sigma_{13}&\sigma_{23}\end{bmatrix}}_{3\times2}\cdot\underbrace{\begin{bmatrix}v_{11}&v_{12}\\v_{21}&v_{22}\end{bmatrix}}_{2\times2}\right\}$$

$$=\frac{1}{2}\operatorname{tr}\left\{\underbrace{\begin{bmatrix}\sum_j\sigma_{1j}^2&\sum_j\sigma_{1j}\sigma_{2j}\\\sum_j\sigma_{2j}\sigma_{1j}&\sum_j\sigma_{2j}^2\end{bmatrix}}_{\text{covariance matrix}}\cdot\begin{bmatrix}v_{11}&v_{12}\\v_{21}&v_{22}\end{bmatrix}\right\}$$

$$=\frac{1}{2}\operatorname{tr}\left[\begin{array}{c|c}v_{11}\cdot\sum_j\sigma_{1j}^2+v_{21}\cdot\sum_j\sigma_{1j}\sigma_{2j}&v_{12}\cdot\sum_j\sigma_{1j}^2+v_{22}\cdot\sum_j\sigma_{1j}\sigma_{2j}\\\hline v_{11}\cdot\sum_j\sigma_{2j}\sigma_{1j}+v_{21}\cdot\sum_j\sigma_{2j}^2&v_{12}\cdot\sum_j\sigma_{2j}\sigma_{1j}+v_{22}\cdot\sum_j\sigma_{2j}^2\end{array}\right]$$

$$=\frac{1}{2}\left\{\underbrace{\left[v_{11}\cdot\sum_j\sigma_{1j}^2+v_{22}\cdot\sum_j\sigma_{2j}^2\right]}_{\text{risk adjustment due to Var}}+\underbrace{\left[v_{21}\cdot\sum_j\sigma_{1j}\sigma_{2j}+v_{12}\cdot\sum_j\sigma_{2j}\sigma_{1j}\right]}_{\text{risk adjustment due to Cov (systematic risk)}}\right\}$$

- Common in finance models and models with systematic risk
- σ may be rectangular or asymmetric, but the covariance matrix $\sigma\sigma^T$ is always symmetric positive semidefinite

Examples: economic models as HJB equations

(*n-independent risks*) However, many macro models fit into the following diagonal-diffusion special case:

$$\mathrm{d}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} \mathrm{d}\, t + \underbrace{\begin{bmatrix} \sigma_{11} & & \\ & \ddots & \\ & & \sigma_{nn} \end{bmatrix}}_{n \times n} \mathrm{d}\begin{bmatrix} W_1 \\ \vdots \\ W_n \end{bmatrix}$$

$$\downarrow\downarrow\downarrow$$

$$\frac{1}{2}\operatorname{tr}\left[\sigma \cdot \sigma^T \cdot \nabla_x^2 v\right] = \frac{1}{2}\sum_{i=1}^{n} \sigma_{ii}^2 \cdot v_{ii}$$

Implications:

- Under this diagonal specification, each state is directly affected by one independent risk source
- In GE, all risks can be expressed as a linear combination of the "final" risk sources (e.g. TFP shock)

Solving policy functions

To solve the policy functions, one can simply derive the first order condition (FOC) of the RHS of the HJB equation.

Example: stochastic neo-classical growth model

$$\rho v(k,z) = \max_c \log(c) + (-\delta k + zk^\alpha - c) \cdot \frac{\partial v}{\partial k} + \theta(\mu - z) \cdot \frac{\partial v}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial z^2}$$

Define objective function:

$$\mathcal{H}\left(c; k, z; \nabla_{k,z} v; \nabla^2_{k,z} v\right) := \log(c) + (-\delta k + zk^\alpha - c) \cdot \frac{\partial v}{\partial k} + \theta(\mu - z) \cdot \frac{\partial v}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial z^2}$$

FOC:

$$\frac{\partial \mathcal{H}}{\partial c} = \frac{1}{c} - \frac{\partial v}{\partial k} = 0$$

$$\implies c = \left(\frac{\partial v}{\partial k}\right)^{-1}$$

Solving policy functions

Compare with the discrete time case:

$$
\begin{aligned}
v(k,z) &= \max_{c} \log(c) + \beta \mathbb{E} v(k',z') \\
\text{s.t.} \quad & c + k' = (1-\delta)k + zk^{\alpha} \\
& z \sim \text{AR}(1) \text{ process}
\end{aligned}
$$

Lagrangian:

$$
\mathcal{L}(c;k,z) := \log(c) + \beta \mathbb{E} v(k',z') + \lambda((1-\delta)k + zk^{\alpha} - c - k')
$$

FOC:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c} &= \frac{1}{c} - \lambda = 0 \\
&\Longrightarrow c = \lambda^{-1}
\end{aligned}
$$

- The Lagrangian multiplier λ is the *shadow value*
- In continuous time, the shadow value is $\frac{\partial v}{\partial k}$

Solving policy functions

Practice 1: solve the optimal inflation rate for the following price setter's problem with Rotemberg price rigidity in continuous time:

$$\rho v(p,y) = \max_{\pi} py \cdot \left(1 - \frac{\varphi}{2}\pi^2\right) + \frac{\partial v}{\partial p} \cdot \pi p + \frac{\partial v}{\partial y} \cdot \theta(\overline{y} - y) + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial y^2}$$

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FOC:

$$\frac{\partial \mathcal{H}}{\partial \pi} = -\varphi p y \pi + p\frac{\partial v}{\partial p} = 0$$

$$\Longrightarrow \pi = \frac{\frac{\partial v}{\partial p}}{\varphi y}$$

Solving policy functions

Practice 2 Solve the policy function for the following household problem with a liquid asset a and an illiquid asset h:

$$\rho v(a,h,z) = \max_{c,I,n} \frac{\left[c^{\alpha}h^{1-\alpha}\right]^{1-\gamma}}{1-\gamma} - \frac{n^{1+\nu}}{1+\nu}$$

$$+\frac{\partial v}{\partial a}\cdot\left(ra+zn-c-\left[I+\frac{\psi}{2}I^2\right]p\cdot h\right)+\frac{\partial v}{\partial h}\cdot Ih+\frac{\partial v}{\partial z}\cdot\theta(\mu-z)+\frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial z^2}$$

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FOC:

$$\frac{\partial \mathcal{H}}{\partial c} = \alpha h^{(1-\alpha)(1-\gamma)}c^{\alpha(1-\gamma)-1} - \frac{\partial v}{\partial a} = 0$$

$$\frac{\partial \mathcal{H}}{\partial I} = -\frac{\partial v}{\partial a}\cdot[1+\psi I]ph + \frac{\partial v}{\partial h}h = 0$$

$$\frac{\partial \mathcal{H}}{\partial n} = -n^{\nu} + \frac{\partial v}{\partial a}z = 0$$

Solving policy functions

which implies:

$$c = \left(\frac{\frac{\partial v}{\partial a}}{\alpha} \right)^{\frac{1}{\alpha(1-\gamma)-1}} h^{-\frac{(1-\alpha)(1-\gamma)}{\alpha(1-\gamma)-1}}$$

$$I = \frac{1}{\psi} \left[\frac{\partial v / \partial h}{\partial v / \partial a} \frac{1}{p} - 1 \right]$$

$$n = \left(\frac{\partial v}{\partial a} z \right)^{\frac{1}{\nu}}$$

- The policy functions are *static* conditional on the current value-function derivatives and current states
- However, consumption dynamics are characterized by the Euler equation, which is central for macroeconomic intuition

Solving policy functions

Dynamic analysis is central to macroeconomic models

- Long-run equilibrium (steady state)
- Inter-temporal trade-off
- Transition dynamics
- ...

This is done by characterizing the dynamics of:

- state variables $x(t)$
- shadow values $\lambda(t)$
- control variables $c(t)$ (Euler equation)

Solving policy functions

- Check the textbook example of the growth model in discrete time:

$$v(k,z) = \max_{c,k'} \log(c) + \beta \mathbb{E}\{v(k',z')|z\}$$
$$\text{s.t.} \quad c + k' = (1-\delta)k + zk^{\alpha}$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial c} = \frac{1}{c} - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial k'} = \beta \mathbb{E} \frac{\partial v'}{\partial k'} - \lambda = 0$$
$$\frac{\partial v'}{\partial k'} = \lambda' \left[(1-\delta) + \alpha z' (k')^{\alpha-1} \right] \text{ (envelope theorem)}$$

Euler equation: $\frac{1}{c} = \beta \mathbb{E}\left\{ \frac{1}{c'} \left[1 - \delta + \alpha z' (k')^{\alpha-1} \right] \right\}$

$\Longrightarrow$ How to analyze the *dynamics* of a continuous time model? e.g.

$$\mathrm{d}\ c(t) = ? \qquad \mathrm{d}\ (\partial v / \partial k) = ?$$

Dynamic analysis in continuous time

Recall the generic discrete time model:

$$v(t,x) = \max_{c_t} \mathbb{E}_t\{u(t,c_t,x_t) + \beta v(t+1,x_{t+1})\}$$
$$\text{s.t. } x_{t+1} = g(t,x_t,c_t); c_t \in \mathbb{C}_t, x_t \in \mathbb{X}_t$$

Dynamics of the model includes the dynamics of:

- state variables $x_t \Longrightarrow$ state equation
- shadow values $\lambda_t \Longrightarrow$ envelope theorem
- control variables $c_t \Longrightarrow$ Euler equation

We typically do:

$$\begin{cases}\text{FOC}\\ \text{Envelope theorem}\end{cases} \Longrightarrow \text{Euler equation}$$

With the dynamics of the model: inter-temporal trade-off; long-run equilibrium (steady state); ...

Dynamic analysis in continuous time

In a continuous time model:

$$0 = \max_{c} \left\{ u(t,c,x) + \frac{\partial v}{\partial t} + \mu(t,x,c)^T \cdot \nabla_x v + \frac{1}{2} \operatorname{tr}(\sigma(t,x,c)\sigma(t,x,c)^T \nabla_x^2 v) - \rho v(t,x) \right\}$$

Dynamics of the model includes the dynamics of:

- state variables $x(t) \Longrightarrow$ state equation d $x(t)$ ✅
- shadow values $\nabla_x v \Longrightarrow$ envelope theorem d $\frac{\partial v}{\partial x_i}$ ❓
- control variables $c(t) \Longrightarrow$ Euler equation d $c(t)$ ❓

$$\begin{cases} \text{FOC} \\ \text{Envelope theorem ?} \end{cases} \Longrightarrow \text{Euler equation}$$

$\Longrightarrow$ *Maximum principle*[1]

[1]The continuous time version of the envelope theorem is available but not easy to derive for the stochastic case, while the maximum principle is much easier to follow.

Dynamic analysis in continuous time

- *Maximum principle*, or Pontryagin's maximum principle (PMP), is a necessary condition for optimality in continuous time control problems

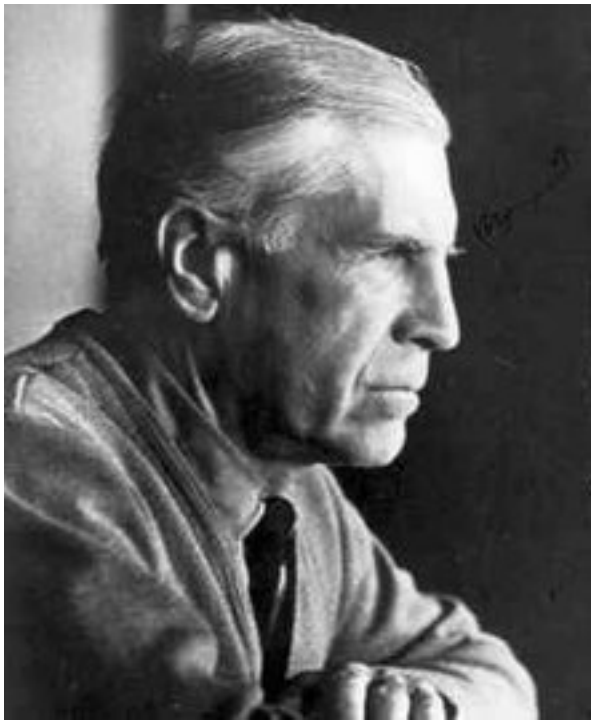


Figure 3: Lev S. Pontryagin (1908-1988)

- Core in stochastic control theory
- We will only cover the maximum principle in the deterministic case

Dynamic analysis in continuous time

To understand what PMP does, let's consider the following deterministic control problem:

$$\max_{c(\cdot)} \int_0^T e^{-\rho t} u(t, c(t), x(t)) \ \mathrm{d}\ t + e^{-\rho T} f(x(T))$$

$$\text{s.t.} \ \ \mathrm{d}\ x(t) = \mu(t, x(t), c(t)) \ \mathrm{d}\ t$$

Define the *(generalized) Hamiltonian*:

$$\mathcal{H}\left(c(t); t, x(t); \{\lambda_i(t)\}_{i=1,\dots,N}\right) := e^{-\rho t} u(t, c(t), x(t)) + \sum_{i=1}^{N} \lambda_i(t) \cdot \mu(t, x(t), c(t))$$

where

- $\lambda_i(t) \in \mathbb{R}$ is the *adjoint variable* or *co-state*, which is the shadow value of the state variable $x_i(t)$

Wait a minute! Does the Hamiltonian look familiar?

Dynamic analysis in continuous time

Recall the objective function of solving the policy functions in HJB:

$$\mathcal{H}\left(c; k, z; \nabla_{k,z} v; \nabla^2_{k,z} v\right) := u(t, c, x) + \mu(t, x, c)^T \cdot \nabla_x v + \frac{1}{2} \operatorname{tr}(\sigma(t, x, c)\sigma(t, x, c)^T \nabla^2_x v)$$

Drop the uncertainty and write each partial derivative:

$$\mathcal{H}\left(c; k, z; \{\partial v / \partial x_i\}_{i=1,\dots,N}\right) := u(t, c, x) + \sum_{i=1}^{N} \frac{\partial v}{\partial x_i} \cdot \mu_i(t, x, c)$$

Then, discount to $t = 0$:

$$\mathcal{H}\left(c; k, z; \{\partial v / \partial x_i\}_{i=1,\dots,N}\right) := e^{-\rho t} u(t, c, x) + \sum_{i=1}^{N} e^{-\rho t} \frac{\partial v}{\partial x_i} \cdot \mu_i(t, x, c)$$

cp. the Hamiltonian in the PMP:

$$\mathcal{H}\left(c(t); t, x(t); \{\lambda_i(t)\}_{i=1,\dots,N}\right) := e^{-\rho t} u(t, c(t), x(t)) + \sum_{i=1}^{N} \lambda_i(t) \cdot \mu(t, x(t), c(t))$$

They share the same form!

Dynamic analysis in continuous time

Intuitively,

$$\lambda_i(t) = e^{-\rho t} \frac{\partial v}{\partial x_i}$$

- maximizing the Hamiltonian $\Longrightarrow$ solving the policy functions
- dynamic of the adjoint variable $\Longleftrightarrow$ dynamics of the shadow value

In addition to finding policy functions, PMP also provides the dynamics of the adjoint variables, which gives the Euler equations when combined with the FOCs of the Hamiltonian

Dynamic analysis in continuous time

When optimal, the *adjoint equation* as a backward differential equation:

$$d\boldsymbol{\lambda}(t) = -\nabla_x \mathcal{H} \text{ d } t$$

or respectively $\forall i = 1, ..., N$:

$$\text{d } \lambda_i(t) = -\frac{\partial \mathcal{H}}{\partial x_i} \text{ d } t = -\left\{ \frac{\partial (e^{-\rho t} \cdot u)}{\partial x_i} + \sum_{j=1}^{N} \lambda_j(t) \cdot \frac{\partial \mu_j}{\partial x_i} \right\} \text{ d } t$$

with a terminal condition (finite horizon):

$$\lambda_i(T) = e^{-\rho T} \frac{\partial f}{\partial x_i}(x^*(T))$$

or a transversality condition (infinite horizon):

$$\lim_{t \to \infty} \lambda_i(t) x_{i(t)} = 0$$

Intuitively consistent with the envelope theorem

Dynamic analysis in continuous time

Example: neo-classical growth model

$$\max_{c(\cdot)} \int_0^\infty e^{-\rho t} \log(c(t)) \text{ d } t$$
$$\text{s.t. } \text{ d } k(t) = [-\delta k(t) + z k(t)^\alpha - c(t)] \text{ d } t$$

Define Hamiltonian:

$$\mathcal{H}(c(t); t, k(t); \lambda_k(t)) := e^{-\rho t} \log(c(t)) + \lambda_k(t) \cdot [-\delta k(t) + z k(t)^\alpha - c(t)]$$

Policy function by maximizing the Hamiltonian:

$$c(t) = \left(e^{\rho t} \lambda_k(t)\right)^{-1} = \left(\frac{\partial v}{\partial k}\right)^{-1}$$

Adjoint equation:

$$\text{d } \lambda_k(t) = -\lambda_k(t) \cdot \left[-\delta + \alpha z k(t)^{\alpha-1}\right] \text{ d } t$$

Dynamic analysis in continuous time

Adjoint equation:

$$\mathrm{d}\ \lambda_k(t) = -\lambda_k(t) \cdot \left[-\delta + \alpha z k(t)^{\alpha-1}\right]\ \mathrm{d}\ t$$

$$\Longrightarrow \frac{\mathrm{d}\ \lambda_k(t)}{\lambda_k(t)} = \left[\delta - \alpha z k(t)^{\alpha-1}\right]\ \mathrm{d}\ t$$

- Interpretation: the present-value shadow value of capital falls faster when the net marginal return of capital is high
- Trick: When $\frac{\mathrm{d}\ x(t)}{x(t)}$ is small, $\mathrm{d}\log(x(t))$ is a good approximation
  - Apply Ito's lemma if needed

$\Longrightarrow$ How to derive the Euler equation then?

Dynamic analysis in continuous time

Recall the FOC of the Hamiltonian:

$$\frac{1}{c(t)} = e^{\rho t}\lambda_k(t)$$

Knowing that $c > 0$, take the log:

$$-\log(c(t)) = \rho t + \log(\lambda_k(t))$$

Take time derivative:

$$-\frac{\mathrm{d}\ c(t)}{c(t)} = \rho\ \mathrm{d}\ t + \frac{\mathrm{d}\ \lambda_k(t)}{\lambda_k(t)}$$

Plugging it into the adjoint equation:

$$\frac{\mathrm{d}\ \lambda_k(t)}{\lambda_k(t)} = -\frac{\mathrm{d}\ c(t)}{c(t)} - \rho\ \mathrm{d}\ t = \left[\delta - \alpha z k(t)^{\alpha-1}\right]\ \mathrm{d}\ t$$

Dynamic analysis in continuous time

Plugging it into the adjoint equation:

$$\frac{\mathrm{d}\ \lambda_k(t)}{\lambda_k(t)} = -\frac{\mathrm{d}\ c(t)}{c(t)} - \rho\ \mathrm{d}\ t = \left[\delta - \alpha z k(t)^{\alpha-1}\right]\ \mathrm{d}\ t$$

Then, we get the Euler equation:

$$\frac{\mathrm{d}\ c(t)}{c(t)} = \left[\alpha z k(t)^{\alpha-1} - \delta - \rho\right]\ \mathrm{d}\ t$$

Interpretation:

- The Euler equation tells us how the consumption growth rate changes over time
- The consumption growth rate reflects the trade-off between the marginal return of capital and the subjective discount rate
- If the net marginal return of capital exceeds the subjective discount rate, consumption grows over time
- If the net marginal return of capital is below the subjective discount rate, consumption falls over time

Dynamic analysis in continuous time

- *(Deterministic) steady state* is the long-run equilibrium of the model
  - No uncertainty (turned off)
  - No growth
  - No allocation change over time

In SS, nothing changes over time:

- $x(t) \equiv \overline{x} \Longrightarrow \mathrm{d}\ x(t) = 0$
- $c(t) \equiv \overline{c} \Longrightarrow \mathrm{d}\ c(t) = 0$

while

- present-value adjoint variables typically decay along a steady state; current-value shadow values such as $\partial \frac{v}{\partial} x$ are constant
- equivalently, $\mathrm{d}\ \frac{\partial v}{\partial x} = 0$ for current-value shadow values

Dynamic analysis in continuous time

Consider the neo-classical growth model example:

$$\begin{aligned}
\mathrm{d}\ k(t) &= \left[-\delta k(t) + z k(t)^{\alpha} - c(t)\right]\ \mathrm{d}\ t \\
\frac{\mathrm{d}\ c(t)}{c(t)} &= \left[\alpha z k(t)^{\alpha-1} - \delta - \rho\right]\ \mathrm{d}\ t
\end{aligned}$$

In SS,

$$\begin{aligned}
\mathrm{d}\ k(t) = 0 &\Longrightarrow -\delta \overline{k} + z \overline{k}^{\alpha} - \overline{c} = 0 \\
\frac{\mathrm{d}\ c(t)}{c(t)} = 0 &\Longrightarrow \alpha z \overline{k}^{\alpha-1} - \delta - \rho = 0
\end{aligned}$$

which gives us the steady state of the model:

$$\begin{aligned}
\overline{k} &= \left(\frac{\alpha z}{\rho + \delta}\right)^{\frac{1}{1-\alpha}} \\
\overline{c} &= -\delta \overline{k} + z \overline{k}^{\alpha}
\end{aligned}$$

Compare with the discrete time version:

$$v(k) = \max_{c,k'} \log(c) + \beta v(k')$$
$$\text{s.t.} \quad c + k' = (1-\delta)k + zk^{\alpha}$$

Euler equation:

$$\frac{c'}{c} = \beta\left[1 - \delta + \alpha z (k')^{\alpha-1}\right]$$

In SS,

$$\overline{c} + \overline{k} = (1-\delta)\overline{k} + z\overline{k}^{\alpha}$$
$$\frac{\overline{c}}{\overline{c}} = 1 = \beta\left[1 - \delta + \alpha z \left(\overline{k}\right)^{\alpha-1}\right]$$

Steady state:

$$\overline{k} = \left(\frac{\alpha z}{\beta^{-1} - (1-\delta)}\right)^{\frac{1}{1-\alpha}}$$
$$\overline{c} = -\delta\overline{k} + z\overline{k}^{\alpha}$$

Dynamic analysis in continuous time

Bridging the two:

Discrete time version:

$$\overline{k} = \left(\frac{\alpha z}{\beta^{-1} - 1 + \delta} \right)^{\frac{1}{1-\alpha}}$$

Continuous time version:

$$\overline{k} = \left(\frac{\alpha z}{\rho + \delta} \right)^{\frac{1}{1-\alpha}}$$

With a discrete time step Δt, the exact relationship is

$$\beta_{\Delta t} = e^{-\rho \Delta t}, \quad \rho = -\frac{\log \beta_{\Delta t}}{\Delta t}$$

When $\rho \Delta t$ is small, the following approximation holds:

$$\beta_{\Delta t}^{-1} - 1 \approx \rho \Delta t$$

Thus the discrete-time and continuous-time steady-state formulas are consistent only after fixing the time unit and matching $\beta_{\Delta t}$ to ρ.

Dynamic analysis in continuous time

Practice 1: solve the necessary condition between ρ and r for existence of a steady state in the following household problem:

$$\rho v(a) = \max_c \frac{c^{1-\gamma}}{1-\gamma} + \frac{\partial v}{\partial a} \cdot (ra - c)$$

Hint:

- Step 1: define Hamiltonian
- Step 2: derive the policy function
- Step 3: derive the adjoint equation
- Step 4: derive the Euler equation
- Step 5: solve the SS of the model

Dynamic analysis in continuous time

Define Hamiltonian:

$$\mathcal{H} := e^{-\rho t} \frac{c^{1-\gamma}}{1-\gamma} + \lambda_a \cdot (ra - c)$$

Solve policy function:

$$c = \left(e^{\rho t} \lambda_a\right)^{-\frac{1}{\gamma}}$$

Adjoint equation:

$$\frac{\mathrm{d}\ \lambda_a}{\lambda_a} = -r\ \mathrm{d}\ t$$

Dynamic analysis in continuous time

Derive Euler equation:

$$\log(c) = -\frac{1}{\gamma}[\rho t + \log(\lambda_a)] \Longrightarrow \frac{\mathrm{d}\ \lambda_a}{\lambda_a} = -\gamma \frac{\mathrm{d}\ c}{c} - \rho\ \mathrm{d}\ t$$

$$\Longrightarrow \frac{\mathrm{d}\ c}{c} = \frac{1}{\gamma}(r - \rho)\ \mathrm{d}\ t$$

If SS exists, then c is constant over time such that $\frac{\mathrm{d}\ c}{c} = 0$

$$\frac{1}{\gamma}(r - \rho)\ \mathrm{d}\ t = 0$$

$$\Longrightarrow r = \rho$$

which is consistent with the intuition of the model, also the standard theory result in the discrete time model

Dynamic analysis in continuous time

Practice 2 (optional): Consider an entrepreneur who can invest in two assets: a liquid capital k of price 1 and an illiquid productive housing wealth h of price p.

$$\rho v(k,h) = \max_{c, I_h} \frac{c^{1-\gamma}}{1-\gamma}$$
$$+\frac{\partial v}{\partial k} \cdot \left[z k^{\alpha} h^{1-\alpha} - \delta_k k - c - \left(I_h + \frac{\psi}{2} I_h^2 \right) p \cdot h \right] + \frac{\partial v}{\partial h} \cdot (I_h - \delta_h) h$$

Question: Solve the house price p that allows the SS to hold

Hint:

- Step 1: define Hamiltonian, solve the policy functions
- Step 2: derive the adjoint equations (and the Euler equations if needed)
- Step 3: combining the above equations to solve the steady state conditions

Dynamic analysis in continuous time

Define Hamiltonian (discounting to $t = 0$):

$$\mathcal{H} := e^{-\rho t}\frac{c^{1-\gamma}}{1-\gamma} + \lambda_k \cdot \left[z k^{\alpha} h^{1-\alpha} - \delta_k k - c - \left(I_h + \frac{\psi}{2} I_h^2 \right) p \cdot h \right] + \lambda_h \cdot (I_h - \delta_h) h$$

Policy functions:

$$c = \left(e^{\rho t} \lambda_k \right)^{-\frac{1}{\gamma}}, \quad I_h = \frac{1}{\psi}\left[\frac{\lambda_h}{\lambda_k}\frac{1}{p} - 1 \right]$$

Adjoint equations:

$$\frac{\mathrm{d}\ \lambda_k}{\lambda_k} = -\left[\alpha z k^{\alpha-1} h^{1-\alpha} - \delta_k \right] \mathrm{d}\ t$$

$$\frac{\mathrm{d}\ \lambda_h}{\lambda_h} = -\frac{\lambda_k}{\lambda_h} \cdot \left[(1-\alpha) z k^{\alpha} h^{-\alpha} - \left(I_h + \frac{\psi}{2} I_h^2 \right) p \right] \mathrm{d}\ t - (I_h - \delta_h)\ \mathrm{d}\ t$$

In SS, the state equations imply

$$\mathrm{d}\ h = 0 \Longrightarrow \overline{I}_h = \delta_h$$

$$\mathrm{d}\ k = 0 \Longrightarrow z\overline{k}^{\alpha}\overline{h}^{1-\alpha} - \delta_k\overline{k} - \overline{c} - \left(\delta_h + \frac{\psi}{2}\delta_h^2\right)p \cdot \overline{h} = 0$$

The present-value adjoint variables decay at the discounting rate ρ:

$$-\rho\ \mathrm{d}\ t = -\left[\alpha z\overline{k}^{\alpha-1}\overline{h}^{1-\alpha} - \delta_k\right]\ \mathrm{d}\ t$$

$$-\rho\ \mathrm{d}\ t = -\frac{\overline{\lambda_k}}{\overline{\lambda_h}} \cdot \left[(1-\alpha)z\overline{k}^{\alpha}\overline{h}^{-\alpha} - \left(\delta_h + \frac{\psi}{2}\delta_h^2\right)p\right]\ \mathrm{d}\ t$$

Notice the SS also satisfies the investment policy function:

$$\overline{I}_h = \delta_h = \frac{1}{\psi}\left[\frac{\overline{\lambda_h}}{\overline{\lambda_k}}\frac{1}{p} - 1\right] \Longrightarrow \frac{\overline{\lambda_h}}{\overline{\lambda_k}} = p(\psi\delta_h + 1)$$

and therefore

$$\frac{\overline{\lambda_k}}{\overline{\lambda_h}} = \frac{1}{[p(\psi\delta_h + 1)]}.$$

Plugging these conditions into the SS adjoint equations:

$$\rho + \delta_k = \alpha z \overline{k}^{\alpha-1} \overline{h}^{1-\alpha}$$

$$\rho = \frac{1}{[p(\psi\delta_h + 1)]} \cdot \left[(1-\alpha) z \overline{k}^{\alpha} \overline{h}^{-\alpha} - \left(\delta_h + \frac{\psi}{2} \delta_h^2 \right) p \right]$$

Let $q := \frac{\overline{k}}{\overline{h}}$. Then

$$q = \left(\frac{\alpha z}{\rho + \delta_k} \right)^{\frac{1}{1-\alpha}}$$

and the house price that supports the SS is

$$p = \frac{(1-\alpha) z q^{\alpha}}{\left[\delta_h + \frac{\psi}{2} \delta_h^2 + \rho(1 + \psi\delta_h) \right]}.$$

This solution is analytical because the first adjoint equation pins down $q = \frac{k}{h}$, and the second adjoint equation then pins down p.

FYI: does this model have unique SS? If yes, how to prove it? If no, why?

FYI: can you derive the counterpart in discrete time?

Preview: HJB, KFE, and equilibrium

For heterogeneous-agent macro models, continuous time often leads to a system of three blocks:

$$\underbrace{\rho v = \max_c \{u(c,x) + \mathcal{L}^{c[v]}\}}_{\text{HJB: individual optimization}}$$

$$\underbrace{\frac{\partial g}{\partial t} = \mathcal{L}^*_{c^*}[g]}_{\text{KFE/Fokker-Planck: distribution dynamics}}$$

$$\underbrace{\mathcal{M}(g, c^*, p) = 0}_{\text{market clearing and prices}}$$

- HJB: solve individual policies $c^*(x)$ given prices
- KFE (Kolmogorov Forward Equation): evolve the cross-sectional distribution $g(x,t)$ given policies
- Equilibrium: prices adjust so markets clear

This preview is not needed for today's derivations, but it explains why HJB/KFE systems are central in HA and HANK models.

(Optional) Numerical methods

- Numerical solutions are necessary for more complex continuous time models
- Nature of the problem: many economically interesting HJBs become constrained PDEs, variational inequalities, or free-boundary problems
- Available methods:
  - Finite difference methods (FDM)
  - Finite element methods (FEM)
  - Finite volume methods (FVM)
  - Spectral methods
  - Machine learning methods (e.g. neural networks)
  - Dynamic programming methods
  - ...
- Currently, FDM is one of the most common methods in economics
  - Pros: straightforward, non-parametric
  - Cons: dimensionality curse, convergence, regular grid, ...

(Optional) Numerical methods

Finite difference method (FDM)

$$0 = \max_c \left\{ u(t,c,x) + \frac{\partial v}{\partial t} + \mu(t,x,c)^T \cdot \nabla_x v + \frac{1}{2} \operatorname{tr}(\sigma(t,x,c)\sigma(t,x,c)^T \nabla_x^2 v) - \rho v(t,x) \right\}$$

- This is a PDE:
  - $\frac{\partial v}{\partial t}$
  - $\nabla_x v = \left(\frac{\partial v}{\partial x_1}, ..., \frac{\partial v}{\partial x_N} \right)^T$
  - $\nabla_x^2 v = \left(\frac{\partial^2 v}{\partial x_i \partial x_j} \right)_{\{i,j=1\}}$
- Idea: use finite difference to approximate the derivatives
  - Forward difference, e.g. $\frac{\partial v}{\partial t} \approx \frac{v(t+\Delta t)-v(t)}{\Delta} t$
  - Backward difference, e.g. $\frac{\partial v}{\partial t} \approx \frac{v(t)-v(t-\Delta t)}{\Delta} t$
  - Central difference, e.g. $\frac{\partial v}{\partial t} \approx \frac{v(t+\Delta t)-v(t-\Delta t)}{2\Delta t}$
- For the 2nd-order derivatives, e.g. $\frac{\partial^2 v}{\partial x_i^2} \approx \frac{v(x+\Delta x_i)-2v(x)+v(x-\Delta x_i)}{\Delta x_i^2}$
- Which kind of approximations to use is not trivial

(Optional) Numerical methods

- Practical FDM needs another full course
- Some concepts that you need to know:
  - Boundary conditions, free boundary problems, and singularities
  - Numerical scheme for FDM
  - FDM convergence analysis
    - Convergence
    - Consistency
    - Stability
    - Monotonicity
  - Barles-Souganidis theorem

Barles, G., & Souganidis, P. E. (1991). Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Analysis*, *4*(3), 271–283.

(Optional) Numerical methods

FYI,

- Lecture notes and code:
  - Benjamin Moll's website: [https://benjaminmoll.com/]
  - esp. applications in heterogeneous agent models and HANK
- Papers:

Kaplan, G., Moll, B., & Violante, G. L. (2018). Monetary policy according to HANK. *American Economic Review*, *108*(3), 697–743.

Frontier topics

- Heterogeneous-agent (HA) models, e.g. HANK
- Perturbation methods for continuous time models, esp. HA models
- High-dimensional methods for continuous time models
- Machine learning methods for continuous time models

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